

CANONICAL BASES FOR THE QUANTUM SUPERGROUPS $U(\mathfrak{gl}_{m|n})$

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ABSTRACT. We give a combinatorial construction for the canonical bases of the \pm -parts of the quantum enveloping superalgebra $U(\mathfrak{gl}_{m|n})$ and discuss their relationship with the Kazhdan-Lusztig bases for the quantum Schur superalgebras $\mathcal{S}(m|n, r)$ introduced in [8]. We will also extend this relationship to the induced bases for simple polynomial representations of $U(\mathfrak{gl}_{m|n})$.

1. INTRODUCTION

The theory of Kazhdan–Lusztig bases for Iwahori–Hecke algebras and its subsequent generalisation by Lusztig to canonical bases for quantum groups and their integrable modules was an important breakthrough in representation theory. Remarkably, this theory can also be approached through Kashiwara’s crystal and global crystal bases and thus results in more applications. For example, it serves as an important motivation for the categorification of quantum enveloping algebras since its geometric construction provides a first model of categorification.

Naturally, generalising the canonical basis (or crystal) theory to the quantum supergroups attracts lots of attention and becomes rather challenging. For example, Benkart–Kang–Kashiwara [2] developed a crystal basis theory for a certain class of representations of the quantum general linear Lie superalgebras; while Clark–Hill–Wang [4] constructed crystal/canonical bases for quantum supergroups with no isotropic odd roots which includes $\mathfrak{so}(1|2n)$ as the only finite type example. More recently, building on the work of Leclerc [14] on quantum shuffles algebras, they [5] established the existence of the canonical basis (called the pseudo-canonical basis loc. cit.) of a quantum supergroup of special type, including the quantum supergroups $U(\mathfrak{gl}_{m|n})$.

In this paper, we will provide a new construction of the canonical basis for the most fundamental quantum supergroup $U(\mathfrak{gl}_{m|n})$, different from the one given in [5]. This approach was motivated by the following. First, canonical bases have been constructed in [8] for quantum Schur superalgebras, which are quotients of $U(\mathfrak{gl}_{m|n})$. Second, the quantum supergroup $U(\mathfrak{gl}_{m|n})$ can be realised as a “limit algebra” of quantum Schur superalgebras [7], which generalises the construction of quantum \mathfrak{gl}_n

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by Beilinson, Lusztig and MacPherson [1]. Thus, it is natural to expect the existence of the canonical basis for (the \pm -parts of) $\mathbf{U}(\mathfrak{gl}_{m|n})$ as a “limit basis” of the canonical bases for quantum Schur superalgebras.

The main discovery in the paper is the identification of the realisation bases of the \pm -parts with PBW type bases. It was observed by Du–Parshall [9] in the quantum \mathfrak{gl}_n case that the BLM realisation bases for the \pm -parts share the same multiplication formula of a basis element by generators as the Ringel–Hall algebra of a linear quiver. In the super case, the nonexistence of Ringel–Hall algebras made us to seek a similar relation directly. Thus, under the realisation isomorphism, we prove in Theorem 4.5 that the realisation basis for the $+$ -part coincides with the PBW type bases considered in [18]. Now the realisation basis has a triangular relation to a certain monomial basis as discovered in the proof of Theorem [7, Th. 8.1] via a similar relation in the quantum Schur superalgebras [7, Th. 7.1]. Thus, we obtain a triangular relation between a monomial basis and a PBW basis. This relation is the key to the existence of the canonical bases (Theorem 5.2) and makes it computable, following an algorithm used in [3]. We will also see in Theorem 5.4 how this canonical basis, as a “limit basis”, is connected to the canonical bases of quantum Schur superalgebras.

The canonical basis for the negative part in the nonsuper case induces nicely canonical bases for simple representations of $\mathbf{U}(\mathfrak{gl}_n)$. However, in the super case, this nice property is no longer true in general. Clark, Hill and Wang conjectured in [5, Conj. 8.9] that the property should hold for $\mathbf{U}(\mathfrak{gl}_{m|1})$ and their polynomial representations. We will prove this part of their conjecture in Corollary 7.12. In general, we will show that, for a simple polynomial representations of $\mathbf{U}(\mathfrak{gl}_{m|n})$, any basis induced from the canonical basis of a quantum Schur superalgebra $\mathcal{S}(m|n, r)$ coincides with the one induced by the canonical basis of the negative part of $\mathbf{U}(\mathfrak{gl}_{m|n})$.

It would be interesting to identify the canonical bases introduced here with the pseudo-canonical bases introduced in [5, (7.3)] and to make a comparison with the canonical basis for the quantum coordinate superalgebra given in [17, 16].

We organise the paper as follows. We will collect the basic theory of quantum Schur superalgebras in §2, including a construction of the canonical basis. We provide in §3 some multiplication formulas of high order in order to construct the Lusztig type form of the \pm -parts and prove that its defining basis is nothing but a PBW type basis in §4. In §5, we construct the canonical bases for the \pm -parts and describe a relation between this basis and that for quantum Schur superalgebras. As examples, we compute the canonical bases for the supergroups $\mathbf{U}(\mathfrak{gl}_{2|1})$ and $\mathbf{U}(\mathfrak{gl}_{2|2})$. Finally, in the last section, we discuss simple polynomial representations of $\mathbf{U}(\mathfrak{gl}_{m|n})$ and relate their bases induced by the canonical bases of $\mathcal{S}(m|n, r)$ and of the negative part of $\mathbf{U}(\mathfrak{gl}_{m|n})$. As an application, we prove the conjecture [5, Conj. 8.9] for polynomial representations.

Throughout, let m, n be nonnegative integers, not both zero. For any integers i, t, s with $0 \leq t \leq s$, define

$$\hat{i} = \begin{cases} 0, & \text{if } 1 \leq i \leq m; \\ 1, & \text{if } m+1 \leq i \leq m+n, \end{cases} \quad \text{and} \quad \begin{bmatrix} t \\ s \end{bmatrix} = \frac{[s]!}{[t]![s-t]!} = \mathbf{v}^{s(t-s)} \begin{bmatrix} t \\ s \end{bmatrix}, \quad (1.0.1)$$

where $[r]! := [1][2] \cdots [r]$ with $[i] = 1 + \mathbf{v}^2 + \cdots + \mathbf{v}^{2(i-1)}$ and $[i] = \begin{bmatrix} i \\ 1 \end{bmatrix} = \frac{\mathbf{v}^i - \mathbf{v}^{-i}}{\mathbf{v} - \mathbf{v}^{-1}}$.

Let \mathbf{v} be an indeterminate and let $\mathbf{v}_a = \mathbf{v}^{(-1)^{\hat{a}}}$ for all $1 \leq a \leq m+n$.

2. CANONICAL BASES FOR QUANTUM SCHUR SUPERALGEBRAS

Let \mathfrak{S}_r be the symmetric group on r letters and let $S = \{(k, k+1) \mid 1 \leq k < r\}$ be the set of basic transpositions. Form the Coxeter system (\mathfrak{S}_r, S) and denote the length function with respect to S by $l : W \rightarrow \mathbb{N}$ and the Bruhat order on \mathfrak{S}_r by \leq .

An N -tuple $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N) \in \mathbb{N}^N$ of non-negative integers is called a composition of r into N parts if $|\lambda| := \sum_i \lambda_i = r$. Let $\Lambda(N, r)$ denote the set of compositions of r into N -parts. A partition π of r is a weakly decreasing sequence $(\pi_1, \pi_2, \dots, \pi_t)$ of nonzero integers. Let $\Pi(r)$ denote the set of partitions of r .

The parabolic (or standard Young) subgroup \mathfrak{S}_λ of \mathfrak{S}_r associated with a composition λ consists of the permutations of $\{1, 2, \dots, r\}$ which leave invariant the following sets of integers

$$R_1^\lambda = \{1, 2, \dots, \lambda_1\}, R_2^\lambda = \{\lambda_1 + 1, \lambda_1 + 2, \dots, \lambda_1 + \lambda_2\}, \dots$$

We will also denote by $\mathcal{D}_\lambda := \mathcal{D}_{\mathfrak{S}_\lambda}$ (resp., \mathcal{D}_λ^+) the set of all distinguished or shortest (resp., longest) coset representatives of the right cosets of \mathfrak{S}_λ in \mathfrak{S}_r . Let $\mathcal{D}_{\lambda\mu} = \mathcal{D}_\lambda \cap \mathcal{D}_\mu^{-1}$, where $\mu \in \Lambda(N, r)$. Then $\mathcal{D}_{\lambda\mu}$ (resp., $\mathcal{D}_{\lambda\mu}^+$) is the set of shortest (resp., longest) \mathfrak{S}_λ - \mathfrak{S}_μ double coset representatives. For $d \in \mathcal{D}_{\lambda\mu}$, the subgroup $\mathfrak{S}_\lambda^d \cap \mathfrak{S}_\mu = d^{-1} \mathfrak{S}_\lambda d \cap \mathfrak{S}_\mu$ is a parabolic subgroup associated with a composition which is denoted by $\lambda d \cap \mu$. In other words, we define

$$\mathfrak{S}_{\lambda d \cap \mu} = \mathfrak{S}_\lambda^d \cap \mathfrak{S}_\mu. \quad (2.0.2)$$

The composition $\lambda d \cap \mu$ can be easily described in terms of the following matrix. Let

$$j(\lambda, d, \mu) = (a_{i,j}), \quad \text{where } a_{i,j} = |R_i^\lambda \cap d(R_j^\mu)|, \quad (2.0.3)$$

be the $N \times N$ matrix associated to the double coset $\mathfrak{S}_\lambda d \mathfrak{S}_\mu$. Then

$$\lambda d \cap \mu = (\nu^1, \nu^2, \dots, \nu^N), \quad (2.0.4)$$

where $\nu^j = (a_{1,j}, a_{2,j}, \dots, a_{N,j})$ is the j th column of A . In this way, the matrix set

$$M(N, r) = \{j(\lambda, d, \mu) \mid \lambda, \mu \in \Lambda(N, r), d \in \mathcal{D}_{\lambda\mu}\}$$

is the set of all $N \times N$ matrices over \mathbb{N} whose entries sum to r . For $A \in M(N, r)$, let

$$\text{ro}(A) := \left(\sum_{j=1}^N a_{1,j}, \dots, \sum_{j=1}^N a_{N,j} \right) = \lambda \quad \text{and} \quad \text{co}(A) := \left(\sum_{i=1}^N a_{i,1}, \dots, \sum_{i=1}^N a_{i,N} \right) = \mu.$$

For nonnegative integers (not both zero) m, n , we often write a composition $\lambda = (\lambda_1, \dots, \lambda_{m+n}) \in \Lambda(m+n, r)$ as $\lambda = (\lambda^{(0)} | \lambda^{(1)})$, where

$$\lambda^{(0)} = (\lambda_1, \dots, \lambda_m), \lambda^{(1)} = (\lambda_{m+1}, \dots, \lambda_{m+n}),$$

to indicate the “even” and “odd” parts of λ and identify $\Lambda(m+n, r)$ with the set

$$\begin{aligned}\Lambda(m|n, r) &= \{\lambda = (\lambda^{(0)}|\lambda^{(1)}) \mid \lambda \in \Lambda(m+n, r)\} \\ &= \bigcup_{r_0+r_1=r} (\Lambda(m, r_0) \times \Lambda(n, r_1)).\end{aligned}$$

Let

$$\begin{aligned}\Lambda^+(m|n, r) &= \{\lambda \in \Lambda(m|n, r) \mid \lambda_1 \geq \cdots \geq \lambda_m, \lambda_{m+1} \geq \cdots \geq \lambda_{m+n}\}, \\ \Lambda(m|n) &= \bigcup_{r \geq 0} \Lambda(m|n, r) = \mathbb{N}^{m+n}, \text{ and } \Lambda^+(m|n) = \bigcup_{r \geq 0} \Lambda^+(m|n, r)\end{aligned}$$

Thus, a parabolic subgroup \mathfrak{S}_λ associated with $\lambda = (\lambda^{(0)}|\lambda^{(1)}) \in \Lambda(m, r_0) \times \Lambda(n, r_1)$ has the even part $\mathfrak{S}_{(\lambda^{(0)}|1^{r_1})}$, briefly denoted by $\mathfrak{S}_{\lambda^{(0)}}$, and the odd part $\mathfrak{S}_{(1^{r_0}|\lambda^{(1)})}$, denoted by $\mathfrak{S}_{\lambda^{(1)}}$.

For $\lambda, \mu \in \Lambda(m|n, r)$, let

$$\mathcal{D}_{\lambda\mu}^\circ = \{d \in \mathcal{D}_{\lambda\mu} \mid \mathfrak{S}_{\lambda^{(0)}}^d \cap \mathfrak{S}_{\mu^{(1)}} = 1, \mathfrak{S}_{\lambda^{(1)}}^d \cap \mathfrak{S}_{\mu^{(0)}} = 1\}. \quad (2.0.5)$$

This set is the super version of the usual $\mathcal{D}_{\lambda\mu}$. Let

$$\begin{aligned}M(m|n, r) &= \{j(\lambda, d, \mu) \mid \lambda, \mu \in \Lambda(m|n, r), d \in \mathcal{D}_{\lambda\mu}^\circ\} \text{ and} \\ M(m|n) &= \bigcup_{r \geq 0} M(m|n, r).\end{aligned} \quad (2.0.6)$$

Actually, from [8, Prop.3.2], if $(a_{i,j}) \in M(m|n, r)$, then $a_{i,j} = 0$ or 1 if $i \leq m < j$ or $j \leq m < i$. We may extend the Bruhat order to $M(m|n, r)$ by setting, for $A = j(\lambda, d, \mu), A' = j(\lambda', d', \mu') \in M(m|n, r)$,

$$A \leq A' \iff \lambda = \lambda', \mu = \mu', \text{ and } d \leq d'. \quad (2.0.7)$$

Let $\mathcal{Z} = \mathbb{Z}[\mathbf{v}, \mathbf{v}^{-1}]$. The Hecke algebra $\mathcal{H} = \mathcal{H}(\mathfrak{S}_r)$ associated to $\mathfrak{S} = \mathfrak{S}_r$ is a free \mathcal{Z} -module with basis $\{T_w; w \in \mathfrak{S}_r\}$ and the multiplication is defined by the rules: for $s \in S$,

$$T_w T_s = \begin{cases} T_{ws}, & \text{if } l(ws) > l(w); \\ (\mathbf{v}^2 - 1)T_w + \mathbf{v}^2 T_{ws}, & \text{otherwise.} \end{cases} \quad (2.0.8)$$

The bar involution on \mathcal{H} is the ring automorphism $\bar{\cdot} : \mathcal{H} \rightarrow \mathcal{H}$ defined by $\bar{\mathbf{v}} = \mathbf{v}^{-1}$ and $\bar{T}_w = (T_{w^{-1}})^{-1}$ for all $w \in \mathfrak{S}$.

For each $\lambda = (\lambda^{(0)}|\lambda^{(1)}) \in \Lambda(m|n, r)$, define

$$x_{\lambda^{(0)}} = \sum_{w \in \mathfrak{S}_{\lambda^{(0)}}} T_w, y_{\lambda^{(1)}} = \sum_{w \in \mathfrak{S}_{\lambda^{(1)}}} (-\mathbf{v}^2)^{-l(w)} T_w.$$

Definition 2.1. Let $\mathfrak{T}(m|n, r) = \bigoplus_{\lambda \in \Lambda(m|n, r)} x_{\lambda^{(0)}} y_{\lambda^{(1)}} \mathcal{H}$. The algebra

$$\mathcal{S}(m|n, r) := \text{End}_{\mathcal{H}}(\mathfrak{T}(m|n, r))$$

is called a *\mathbf{v} -Schur superalgebra* over \mathcal{Z} on which the \mathbb{Z}_2 -graded structure is induced from the \mathbb{Z}_2 -graded structure on $\mathfrak{T}(m|n, r)$ defined by

$$\mathfrak{T}(m|n, r)_i = \bigoplus_{\substack{\lambda \in \Lambda(m|n, r) \\ |\lambda^{(1)}| \equiv i \pmod{2}}} x_{\lambda^{(0)}} y_{\lambda^{(1)}} \mathcal{H}_R \quad (i = \bar{0}, \bar{1}).$$

Following [8], define, for $\lambda, \mu \in \Lambda(m|n, r)$ and $d \in \mathcal{D}_{\lambda\mu}^\circ$,

$$T_{\mathfrak{S}_\lambda d \mathfrak{S}_\mu} := \sum_{\substack{w_0 w_1 \in \mathfrak{S}_\mu \cap \mathcal{D}_{\lambda d \cap \mu}, \\ w_0 \in \mathfrak{S}_{\mu^{(0)}}, w_1 \in \mathfrak{S}_{\mu^{(1)}}}} (-\mathbf{v}^2)^{-\ell(w_1)} x_{\lambda^{(0)}} y_{\lambda^{(1)}} T_d T_{w_0} T_{w_1}.$$

There exists \mathcal{H} -homomorphism

$$\phi_{\lambda\mu}^d(x_{\alpha^{(0)}} y_{\alpha^{(1)}}) = \delta_{\mu, \alpha} T_{\mathfrak{S}_\lambda d \mathfrak{S}_\mu} h, \forall \alpha \in \Lambda(m|n, r), h \in \mathcal{H}.$$

If $A = j(\lambda, d, \mu)$, denote $\phi_A := \phi_{\lambda\mu}^d$. The following result is given in [8, 5.8].

Lemma 2.2. *The set $\{\phi_A \mid A \in M(m|n, r)\}$ forms a \mathcal{Z} -basis for $\mathcal{S}(m|n, r)$.*

In order to define the canonical basis, we use the normalised basis $\{\varphi_A \mid A \in M(m|n, r)\}$ defined as follow.

For $\lambda, \mu \in \Lambda(m|n, r)$ and $d \in \mathcal{D}_{\lambda\mu}^\circ$, set d^* (resp. $*d$) to be the longest element in the double coset $\mathfrak{S}_{\lambda^{(0)}} d \mathfrak{S}_{\mu^{(0)}} (resp. \mathfrak{S}_{\lambda^{(1)}} d \mathfrak{S}_{\mu^{(1)}})$. If $A = j(\lambda, d, \mu)$, by [8, (6.0.2)], let¹

$$\mathcal{T}_A = \mathbf{v}^{-l(d^*) + l(*d) - l(d)} T_A \quad \text{and} \quad \varphi_A = \mathbf{v}^{-l(d^*) + l(*d) - l(d) + l(w_{0, \mu^{(0)}}) - l(w_{0, \mu^{(1)}})} \phi_A,$$

where $w_{0, \lambda}$ denotes the longest element in \mathfrak{S}_λ .

The *bar involution* on \mathcal{H} can be extended to the quantum Schur superalgebra

$$\bar{\cdot} : \mathcal{S}(m|n, r) \longrightarrow \mathcal{S}(m|n, r) \text{ satisfying } \bar{\mathbf{v}} = \mathbf{v}^{-1}, \overline{\varphi_A} = \sum_{B \leq A} r_{B,A} \varphi_B, \quad (2.2.1)$$

where $r_{B,A}$ is defined by $\overline{\mathcal{T}_A} = \sum_{B \leq A} r_{B,A} \mathcal{T}_B$.

Let

$$[A] = (-1)^{\hat{A}} \varphi_A \quad \text{where} \quad \hat{A} = \sum_{\substack{m < k < i \leq m+n \\ 1 \leq j < l \leq m+n}} a_{i,j} a_{k,l}. \quad (2.2.2)$$

Recall from [8, §8] that the \mathbf{v} -Schur superalgebra $\mathcal{S}(m|n, r)$ can also be defined as the endomorphism algebra $\text{End}_{\mathcal{H}}(V(m|n)^{\otimes r})$ of the tensor space $V(m|n)^{\otimes r}$; see Corollary 8.4 there. Here $V(m|n)$ is a free \mathcal{Z} -module of rank $m+n$ with basis v_1, v_2, \dots, v_{m+n} , where v_1, v_2, \dots, v_m are even and v_{m+1}, \dots, v_{m+n} are odd. Its tensor product $V(m|n)^{\otimes r}$ has the basis $\{v_{\mathbf{i}} := v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_r}\}_{\mathbf{i} \in I(m|n, r)}$ where

$$I(m|n, r) = \{\mathbf{i} = (i_1, i_2, \dots, i_r) \mid 1 \leq i_j \leq m+n, \forall j\}.$$

The place permutation (right) action of the symmetric group \mathfrak{S}_r acts on $I(m|n, r)$ induces right \mathcal{H} -module structure on $V(m|n)^{\otimes r}$; see [7, (1.1.1)]. For $A = j(\lambda, d, \mu) \in$

¹We have corrected some typos given in [8, (6.2.1)].

$M(m|n, r)$, let $\zeta_A \in \text{End}_{\mathcal{H}}(V(m|n)^{\otimes r})$ be defined by

$$\zeta_A(v_\mu) = (v_\mu)N_{\mathfrak{S}_r, \mathfrak{S}_\lambda^d \cap \mathfrak{S}_\mu}(e_{\mu, \lambda d}) = \sum_{w \in \mathcal{D}_{\lambda d \cap \mu} \cap \mathfrak{S}_\mu} (-\mathbf{q})^{-l(w_1)} (v_{\mathbf{i}_\lambda d}) T_w,$$

where $N_{\mathfrak{S}_r, \mathfrak{S}_\lambda^d \cap \mathfrak{S}_\mu}(e_{\mu, \lambda d})$ is the relative norm defined in [7, (1.1.2)], $v_\mu = v_{\mathbf{i}_\mu}$ with

$$\mathbf{i}_\mu = (\underbrace{1, \dots, 1}_{\mu_1}, \underbrace{2, \dots, 2}_{\mu_2}, \dots, \underbrace{m+n, \dots, m+n}_{\mu_{m+n}}) = (1^{\lambda_1}, 2^{\lambda_2}, \dots, (m+n)^{\lambda_{m+n}})$$

and w_1 is an “odd” component of $w = w_0 w_1$ with $w_i \in \mathfrak{S}_{\mu(i)}$. Following [7, (4.2.1)], let

$$\xi_A = \mathbf{v}^{-d(A)} \zeta_A \quad \text{where } d(A) = \sum_{i>k, j<l} a_{i,j} a_{k,l} + \sum_{j<l} (-1)^{\hat{i}} a_{i,j} a_{i,l}. \quad (2.2.3)$$

We have the following identification between the bases $\{[A]\}$ and $\{\xi_A\}$.

Lemma 2.3. *By identifying $\mathcal{S}(m|n, r)$ with $\text{End}_{\mathcal{H}}(V(m|n)^{\otimes r})$ under the isomorphism given in [8, Cor. 8.4], we have $\xi_A = [A] = (-1)^{\hat{A}} \varphi_A$ for all $A \in M(m|n, r)$.*

Proof. By [8, Prop.8.3], the map $f : V_R(m|n)^{\otimes r} \rightarrow \mathfrak{T}_R(m|n, r)$ sending $(-1)^{\hat{d}} v_{\mathbf{i}_\lambda d}$ to $x_{\lambda(0)} y_{\lambda(1)} T_d$, for any $\lambda \in \Lambda(m|n, r)$ and $d \in \mathcal{D}_\lambda$, is an \mathcal{H} -module isomorphism. Here $\hat{d} = \sum_{k=1}^{r-1} \sum_{k<l, i_k > i_l} \hat{i}_k \hat{i}_l$ for $\mathbf{i} = \mathbf{i}_\lambda d$. It is direct to check that $\phi_A \circ f = (-1)^{\hat{d}} f \circ \zeta_A$.

Now, for $A \in M(m|n, r)$ with $A = j(\lambda, d, \mu)$, we have by Remark [7, Remark 4.3], $d(A) = l(d^*) - l(*d) + l(d) - l(w_{0, \mu(0)}) + l(w_{0, \mu(1)})$ and, by [7, Lem. 2.3], $\hat{A} = \hat{d}$. The assertion follows immediately. \square

In [8], a canonical basis $\{\Theta_A\}_A$ is constructed relative to the basis $\{\varphi_A\}_A$ and the bar involution defined in [8, Th. 6.3]. By the lemma above, the canonical basis $\{\Xi_A\}_A$ relative to the basis $\{[A]\}_A$ and the same bar involution can be similarly defined.

Corollary 2.4. *Let $\mathcal{C}_r = \{\Xi_A \mid A \in M(m|n, r)\}$ (resp., $\{\Theta_A \mid A \in M(m|n, r)\}$) be the canonical basis defined relative to basis $\{[A]\}_A$ (resp., $\{\varphi_A\}_A$), the bar involution (2.2.1), and the Bruhat order \leq . Then $\Xi_A = (-1)^{\hat{A}} \Theta_A$.*

Proof. Since $\{\Xi_A\}_A$ (resp., $\{\Theta_A\}_A$) is the unique basis satisfying $\overline{\Xi_A} = \Xi_A$ (resp., $\overline{\Theta_A} = \Theta_A$) and

$$\Xi_A - [A] \in \sum_{B < A} \mathbf{v}^{-1} \mathbb{Z}[\mathbf{v}^{-1}][B] \quad (\text{resp., } \Theta_A - \varphi_A \in \sum_{B < A} \mathbf{v}^{-1} \mathbb{Z}[\mathbf{v}^{-1}]\varphi_B).$$

If we write $\Theta_A = \varphi_A + \sum_{B < A} p_{B,A} \varphi_B$, then, by the lemma above,

$$(-1)^{\hat{A}} \Theta_A = [A] + \sum_{B < A} (-1)^{\hat{A} + \hat{B}} p_{B,A} [B] \quad \text{and} \quad \overline{(-1)^{\hat{A}} \Theta_A} = (-1)^{\hat{A}} \Theta_A.$$

The uniqueness forces $\Xi_A = (-1)^{\hat{A}} \Theta_A$. \square

We will discuss a PBW type basis for $\mathcal{S}(m|n, r)$ at the end of §5.

3. MULTIPLICATION FORMULAS AND A STABILISATION PROPERTY

We first record the following multiplication formulas discovered in [7, Props. 4.4-5]. For a fixed matrix $A \in M(m|n, r)$, $h \in [1, m+n)$ and $p \geq 1$, let

$$\begin{aligned} U_p &= U_p(h, \text{ro}(A)) = \text{diag}(\text{ro}(A) - p\mathbf{e}_{h+1}) + pE_{h,h+1} \in M(m|n, r) \\ L_p &= L_p(h, \text{ro}(A)) = \text{diag}(\text{ro}(A) - p\mathbf{e}_h) + pE_{h+1,h} \in M(m|n, r). \end{aligned} \quad (3.0.1)$$

Proposition 3.1. *Maintain the notation above. The following multiplication formulas hold in the \mathbf{v} -Schur superalgebra $\mathcal{S}(m|n, r)$ over \mathcal{Z} :*

If $\mathbf{h} \neq \mathbf{m}$, then

$$\begin{aligned} (1^+) \quad [U_p][A] &= \sum_{\substack{\nu \in \Lambda(m|n, p) \\ \nu \leq \text{row}_{h+1}(A)}} \mathbf{v}_h^{f_h(\nu, A)} \prod_{k=1}^{m+n} \overline{\left[\begin{smallmatrix} a_{h,k} + \nu_k \\ \nu_k \end{smallmatrix} \right]} \mathbf{v}_h^2 [A + \sum_l \nu_l (E_{h,l} - E_{h+1,l})], \\ (1^-) \quad [L_p][A] &= \sum_{\substack{\nu \in \Lambda(m|n, p) \\ \nu \leq \text{row}_h(A)}} \mathbf{v}_{h+1}^{g_h(\nu, A)} \prod_{k=1}^{m+n} \overline{\left[\begin{smallmatrix} a_{h+1,k} + \nu_k \\ \nu_k \end{smallmatrix} \right]} \mathbf{v}_{h+1}^2 [A - \sum_l \nu_l (E_{h,l} - E_{h+1,l})], \end{aligned}$$

where $\lambda \leq \mu \iff \lambda_i \leq \mu_i \ \forall i$,

$$\begin{aligned} f_h(\nu, A) &= \sum_{j \geq t} a_{h,j} \nu_t - \sum_{j > t} a_{h+1,j} \nu_t + \sum_{t < t'} \nu_t \nu_{t'}, \text{ and} \\ g_h(\nu, A) &= \sum_{j \leq t} a_{h+1,j} \nu_t - \sum_{j < t} a_{h,j} \nu_t + \sum_{t < t'} \nu_t \nu_{t'}. \end{aligned} \quad (3.1.1)$$

If $\mathbf{h} = \mathbf{m}$, then $[U_p][A] = 0 = [L_p][A]$ for all $p > 1$ and

$$\begin{aligned} (2^+) \quad [U_1][A] &= \sum_{\substack{k \in [1, m+n] \\ a_{m+1,k} \geq 1}} (-1)^{\sum_{i>m, j<k} a_{i,j}} \mathbf{v}_m^{f_m(\mathbf{e}_k, A)} \overline{[a_{m,k} + 1]} \mathbf{v}_m^2 [A + E_{m,k} - E_{m+1,k}]; \\ (2^-) \quad [L_1][A] &= \sum_{\substack{k \in [1, m+n] \\ a_{m,k} \geq 1}} (-1)^{\sum_{i>m, j<k} a_{i,j}} \mathbf{v}_{m+1}^{g_m(\mathbf{e}_k, A)} \overline{[a_{m+1,k} + 1]} \mathbf{v}_{m+1}^2 [A - E_{m,k} + E_{m+1,k}], \end{aligned}$$

where

$$f_m(\mathbf{e}_k, A) = \sum_{j \geq k} a_{m,j} + \sum_{j > k} a_{m+1,j} \text{ and } g_m(\mathbf{e}_k, A) = \sum_{j \leq k} a_{m+1,j} + \sum_{j < k} a_{m,j}. \quad (3.1.2)$$

We now describe a stabilisation property from these formulas which is the key to a realisation of the supergroup $U(\mathfrak{gl}_{m|n})$.

For $A = (a_{i,j}) \in M(m|n, r)$, let

$$\bar{A} = \sum_{\substack{m+n \geq i > m \geq k \geq 1 \\ m < j < l \leq m+n}} a_{i,j} a_{k,l}. \quad (3.1.3)$$

Note that this number is different from the number \hat{A} defined in (2.2.2).

Consider the set of matrices with zero diagonal:

$$M(m|n)^\pm = \{A = (a_{i,j}) \in M(m|n) \mid a_{i,i} = 0, 1 \leq i \leq m+n\}.$$

Define $M(m|n)^+$ (resp., $M(m|n)^-$) as the subset of upper (resp., lower) triangular elements in $M(m|n)^\pm$. For $A \in M_{m+n}(\mathbb{Z})$ and $\mathbf{j} = (j_1, j_2, \dots, j_{m+n}) \in \mathbb{Z}^{m+n}$, define

$$A(\mathbf{j}, r) = \begin{cases} \sum_{\lambda \in \Lambda(m|n, r-|A|)} (-1)^{\overline{A+\text{diag}(\lambda)}} \mathbf{v}^{\lambda \cdot \mathbf{j}} [A + \text{diag}(\lambda)], & \text{if } A \in M(m|n)^\pm, |A| \leq r; \\ 0, & \text{otherwise,} \end{cases} \quad (3.1.4)$$

where $\cdot = \cdot_s$ denotes the super (or signed) “dot product”:

$$\lambda \cdot \mathbf{j} = \sum_{i=1}^{m+n} (-1)^{\widehat{i}} \lambda_i j_i = \lambda_1 j_1 + \dots + \lambda_m j_m - \lambda_{m+1} j_{m+1} - \dots - \lambda_{m+n} j_{m+n}. \quad (3.1.5)$$

We have the following stabilisation property.

Proposition 3.2 ([7, 5.3, 5.6]). *For all $r \geq 0$, the set*

$$\mathcal{L}_r = \{A(\mathbf{j}, r) \mid A \in M(m|n)^\pm, \mathbf{j} \in \mathbb{Z}^{m+n}\}$$

spans the \mathbf{v} -Schur superalgebra $\mathcal{S}(m|n, r)$ over $\mathbb{Q}(\mathbf{v})$. Moreover, $E_{h, h+1}(\mathbf{0}, r)A(\mathbf{j}, r)$ and $E_{h+1, h}(\mathbf{0}, r)A(\mathbf{j}, r)$ can be written as a linear combination of certain (linearly independent) elements of \mathcal{L}_r with coefficients independent of r for all $r \geq |A|$.

We write explicitly a special case of these multiplication formulas.

Lemma 3.3 ([7, 6.2]). *For fixed $A = (a_{i,j}) \in M(m|n)^+$ and $1 \leq h < m+n$, let*

$$\begin{aligned} \sigma(k) &= \sigma_A(k) := \sum_{i \leq m, j > k} a_{i,j} \\ f(h, k) &= f_A(h, k) := \sum_{j \geq k} a_{h,j} - (-1)^{\delta_{m,h}} \sum_{j > k} a_{h+1,j}, \end{aligned} \quad (1 \leq k \leq m+n). \quad (3.3.1)$$

The following multiplication formulas hold in $\mathcal{S}(m|n, r)$ for all $r \geq |A|$:

$$\begin{aligned} E_{h, h+1}(\mathbf{0}, r)A(\mathbf{0}, r) &= (-1)^{\sigma(h+1)\delta_{h,m}} \mathbf{v}_h^{f(h, h+1)} \overline{\llbracket a_{h, h+1} + 1 \rrbracket}_{\mathbf{v}_h^2} (A + E_{h, h+1})(\mathbf{0}, r) \\ &+ \sum_{k > h+1, a_{h+1, k} \geq 1} (-1)^{\sigma(k)\delta_{h,m}} \mathbf{v}_h^{f(h, k)} \overline{\llbracket a_{h, k} + 1 \rrbracket}_{\mathbf{v}_h^2} (A + E_{h, k} - E_{h+1, k})(\mathbf{0}, r); \end{aligned}$$

We now generalise this property to the higher order situation. By Proposition 3.1(2 $^\pm$), we only need to consider the $h \neq m$ case.

Lemma 3.4. *Let $A = (a_{i,j}) \in M(m|n)^+$ and $h \in [1, m+n]$ with $h \neq m$ and let p be any positive integer. Then, for all $r \geq |A|$, the following multiplication formula holds in $\mathcal{S}(m|n, r)$:*

$$(pE_{h, h+1})(\mathbf{0}, r)A(\mathbf{0}, r) = \sum_{\substack{\nu \in \Lambda(m|n, p) \\ \nu - \nu_{h+1} \mathbf{e}_{h+1} \leq \text{row}_{h+1}(A)}} \mathbf{v}_h^{f_h(\nu, A)} \prod_{k=1}^{m+n} \overline{\llbracket a_{h, k} + \nu_k \rrbracket}_{\nu_k} A^{[\nu]}(\mathbf{0}, r),$$

where $f_h(\nu, A)$ is defined in (3.1.1) and $A^{[\nu]} = A + \nu_{h+1} E_{h, h+1} + \sum_{l \neq h+1} \nu_l (E_{h, l} - E_{h+1, l})$.

Proof. For notational simplicity, let

$$\Lambda' := \Lambda(m|n, r - |A|) \text{ and } A^\mu = (a_{i,j}^\mu) := A + \text{diag}(\mu) \quad \forall \mu \in \Lambda'.$$

By definition and Proposition 3.1, the left hand side becomes

$$\begin{aligned} \text{LHS} &= \sum_{\lambda \in \Lambda(m|n, r-p)} [pE_{h,h+1} + \text{diag}(\lambda)] \sum_{\mu \in \Lambda'} (-1)^{\overline{A^\mu}} [A^\mu] \\ &= \sum_{\mu \in \Lambda'} (-1)^{\overline{A^\mu}} \sum_{\substack{\nu \in \Lambda(m|n, p) \\ \nu \leq \text{row}_{h+1}(A^\mu)}} \mathbf{v}_h^{f_h(\nu, A^\mu)} \prod_{k=1}^{m+n} \overline{\begin{bmatrix} a_{h,k}^\mu + \nu_k \\ \nu_k \end{bmatrix}}_{\mathbf{v}_h^2} [A^\mu + \sum_l \nu_l (E_{h,l} - E_{h+1,l})] \\ &= \sum_{\mu \in \Lambda'} \sum_{\substack{\nu \in \Lambda(m|n, p) \\ \nu \leq \text{row}_{h+1}(A^\mu)}} \mathbf{v}_h^{f_h(\nu, A)} \prod_{k=1}^{m+n} \overline{\begin{bmatrix} a_{h,k} + \nu_k \\ \nu_k \end{bmatrix}}_{\mathbf{v}_h^2} (-1)^{\overline{A^\mu}} [A^\mu + \sum_l \nu_l (E_{h,l} - E_{h+1,l})], \end{aligned}$$

where the last equality is seen as follows. Since $A \in M(m|n)^+$, the first h entries of $\text{row}_{h+1}(A^\mu)$ are zero. Hence, $\nu = (\underbrace{0, \dots, 0}_h, \nu_{h+1}, \dots)$. Thus,

$$\begin{aligned} f_h(\nu, A^\mu) &= \sum_{j \geq t} a_{h,j}^\mu \nu_t - \sum_{j > t} a_{h+1,j}^\mu \nu_t + \sum_{t < t'} \nu_t \nu_{t'} \\ &= \sum_{j \geq t > h} a_{h,j}^\mu \nu_t - \sum_{j > t > h} a_{h+1,j}^\mu \nu_t + \sum_{t < t'} \nu_t \nu_{t'} \\ &= \sum_{j \geq t > h} a_{h,j} \nu_t - \sum_{j > t > h} a_{h+1,j} \nu_t + \sum_{t < t'} \nu_t \nu_{t'} = f_h(\nu, A) \end{aligned}$$

and

$$\prod_{k=1}^{m+n} \overline{\begin{bmatrix} a_{h,k}^\mu + \nu_k \\ \nu_k \end{bmatrix}}_{\mathbf{v}_h^2} = \prod_{k > h}^{m+n} \overline{\begin{bmatrix} a_{h,k} + \nu_k \\ \nu_k \end{bmatrix}}_{\mathbf{v}_h^2} = \prod_{k=1}^{m+n} \overline{\begin{bmatrix} a_{h,k} + \nu_k \\ \nu_k \end{bmatrix}}_{\mathbf{v}_h^2}.$$

Let $\Lambda''(\mu) = \{\nu \in \Lambda(m|n, p) \mid \nu \leq \text{row}_{h+1}(A^\mu)\}$. Then, for $p' = \min(p, r - |A|)$,

$$\mathcal{X} := \{(\mu, \nu) \mid \mu \in \Lambda', \nu \in \Lambda''(\mu)\} = \bigcup_{a=0}^{p'} \mathcal{X}_a,$$

where the union is disjoint and

$$\mathcal{X}_a = \{(\mu, \nu) \in \mathcal{X} \mid \nu_{h+1} = a\} = \{(\mu, \nu) \mid \mu \in \Lambda', \mu_{h+1} \geq a, \nu \in \Lambda''(\mu), \nu_{h+1} = a\}.$$

Clearly there is a bijection between sets

$$\{\nu \in \Lambda''(\mu) \mid \nu_{h+1} = a\} \text{ and } \{\nu' \in \Lambda(m|n, p-a) \mid \nu' \leq \text{row}_{h+1}(A)\},$$

where $\nu' = \nu - \nu_{h+1} \mathbf{e}_{h+1} = (\underbrace{0, \dots, 0}_{h+1}, \nu_{h+1}, \dots)$. Moreover, since $h \neq m$, by (3.1.3),

$$\overline{A^\mu} = \overline{A^\mu + \sum_l \nu_l (E_{h,l} - E_{h+1,l})} = \overline{A^{[\nu]} + \text{diag}(\lambda - \mu_{h+1} \mathbf{e}_{h+1})}.$$

Continuing our computation by swapping the summations yields

$$\begin{aligned}
\text{LHS} &= \sum_{a=0}^{p'} \sum_{(\mu, \nu) \in \mathcal{X}_a} \mathbf{v}_h^{f_h(\nu, A)} \prod_{k=1}^{m+n} \overline{\left[\begin{matrix} a_{h,k} + \nu_k \\ \nu_k \end{matrix} \right]}_{\mathbf{v}_h^2} (-1)^{\overline{A^\mu}} [A^\mu + \sum_l \nu_l (E_{h,l} - E_{h+1,l})] \\
&= \sum_{a=0}^{p'} \sum_{\substack{\mu \in \Lambda', \\ \mu_{h+1} \geq a}} \sum_{\substack{\nu' \in \Lambda(m|n, p-a) \\ \nu' \leq \text{row}_{h+1}(A)}} \mathbf{v}_h^{f_h(\nu, A)} \prod_{k=1}^{m+n} \overline{\left[\begin{matrix} a_{h,k} + \nu_k \\ \nu_k \end{matrix} \right]}_{\mathbf{v}_h^2} (-1)^{\overline{A^\mu}} \\
&\quad \cdot [A^\mu + a(E_{h,h+1} - E_{h+1,h+1}) + \sum_{l \neq h+1} \nu_l (E_{h,l} - E_{h+1,l})],
\end{aligned}$$

where $\nu = \nu' + a\mathbf{e}_{h+1}$,

$$\begin{aligned}
&= \sum_{a=0}^{p'} \sum_{\substack{\nu' \in \Lambda(m|n, p-a), \\ \nu' \leq \text{row}_{h+1}(A)}} \mathbf{v}_h^{f_h(\nu, A)} \prod_{k=1}^{m+n} \overline{\left[\begin{matrix} a_{h,k} + \nu_k \\ \nu_k \end{matrix} \right]}_{\mathbf{v}_h^2} \\
&\quad \cdot \sum_{\substack{\mu \in \Lambda' \\ \mu_{h+1} \geq a}} (-1)^{\overline{A^\mu}} [A + aE_{h,h+1} + \sum_{l \neq h+1} \nu_l (E_{h,l} - E_{h+1,l}) + \text{diag}(\mu - a\mathbf{e}_{h+1})] \\
&= \sum_{\substack{\nu \in \Lambda(m|n, p) \\ \nu - \nu_{h+1}\mathbf{e}_{h+1} \leq \text{row}_{h+1}(A)}} \mathbf{v}_h^{f_h(\nu, A)} \prod_{k=1}^{m+n} \overline{\left[\begin{matrix} a_{h,k} + \nu_k \\ \nu_k \end{matrix} \right]}_{\mathbf{v}_h^2} A^{[\nu]}(\mathbf{0}, r) = \text{RHS}.
\end{aligned}$$

□

4. THE REALISATION OF A PBW BASIS FOR $\mathbf{U}(\mathfrak{gl}_{m|n})$

We now use the stabilisation property developed in Lemmas 3.3 and 3.4 to give a realisation of the Lusztig form $U_{\mathbb{Z}}^{\pm}(\mathfrak{gl}_{m|n})$ of the \pm -parts of $\mathbf{U}(\mathfrak{gl}_{m|n})$ and introduce the canonical basis for $U_{\mathbb{Z}}^{\pm}(\mathfrak{gl}_{m|n})$. We first recall the realisation of $\mathbf{U}(\mathfrak{gl}_{m|n})$ via the stabilisation property mentioned in Proposition 3.2.

Recall also the definition of the super commutator on homogeneous elements of a superalgebra with parity function $\widehat{}$:

$$[X, Y] = XY - (-1)^{\widehat{X}\widehat{Y}} YX.$$

Definition 4.1 ([18]). The quantum supergroup $\mathbf{U} = \mathbf{U}(\mathfrak{gl}_{m|n})$ is the superalgebra over $\mathbb{Q}(\mathbf{v})$ with

even generators: $K_a, K_a^{-1}, E_h, F_h, 1 \leq a, h \leq m+n, h \neq m, m+n$, and

odd generators: E_m, F_m

which satisfy the following relations:

- (QS1) $K_a K_a^{-1} = 1, K_a K_b = K_b K_a$;
- (QS2) $K_a E_h = \mathbf{v}^{e_a \cdot s \alpha_h} E_h K_a, K_a F_h = \mathbf{v}^{-e_a \cdot s \alpha_h} F_h K_a$;
- (QS3) $[E_h, F_k] = \delta_{h,k} \frac{K_h K_{h+1}^{-1} - K_h^{-1} K_{h+1}}{\mathbf{v}_h - \mathbf{v}_h^{-1}},$
- (QS4) $E_h E_k = E_k E_h, F_h F_k = F_k F_h$, if $|k - h| > 1$;

(QS5) For $h \neq m$ and $|h - k| = 1$,

$$\begin{aligned} E_h^2 E_k - (\mathbf{v}_h + \mathbf{v}_h^{-1}) E_h E_k E_h + E_k E_h^2 &= 0, \\ F_h^2 F_k - (\mathbf{v}_h + \mathbf{v}_h^{-1}) F_h E_k F_h + F_k F_h^2 &= 0, \end{aligned}$$

(QS6) $E_m^2 = F_m^2 = [E_m, E_{m-1, m+2}] = [F_m, E_{m+2, m-1}] = 0$, where $E_{m-1, m+2}$, $E_{m+2, m-1}$ denote respectively the elements

$$\begin{aligned} E_{m-1} E_m E_{m+1} - \mathbf{v} E_{m-1} E_{m+1} E_m - \mathbf{v}^{-1} E_m E_{m+1} E_{m-1} + E_{m+1} E_m E_{m-1}, \\ F_{m+1} F_m F_{m-1} - \mathbf{v}^{-1} F_m F_{m+1} F_{m-1} - \mathbf{v} F_{m-1} F_{m+1} F_m + F_{m-1} F_m F_{m+1}. \end{aligned}$$

Clearly, \mathbf{U} admits a $\mathbb{Q}(\mathbf{v})$ -algebra anti-involution (i.e., anti-automorphism of order two):

$$\tau : \mathbf{U} \longrightarrow \mathbf{U}, \quad E_h \longmapsto F_h, F_h \longmapsto E_h, K_i^{\pm 1} \longmapsto K_i^{\pm 1}. \quad (4.1.1)$$

The *quantum root vectors* $E_{a,b}$, for $a, b \in [1, m+n]$ with $|a - b| \geq 1$, are defined by recursively setting $E_{h, h+1} = E_h$, $E_{h+1, h} = F_h$, and

$$E_{a,b} = \begin{cases} E_{a,c} E_{c,b} - \mathbf{v}_c^{-1} E_{c,b} E_{a,c}, & \text{if } a < b; \\ E_{a,c} E_{c,b} - \mathbf{v}_c E_{c,b} E_{a,c}, & \text{if } a > b, \end{cases} \quad (4.1.2)$$

where c can be taken to be an arbitrary index strictly between a and b , and $E_{a,b}$ is homogeneous of degree $\widehat{E}_{a,b} := \widehat{a} + \widehat{b}$. We remarks that τ does not send the positive root vector to negative root vectors, i.e., $\tau(E_{a,b}) \neq E_{b,a}$ for all $a + 1 < b$.

Let $\Pi = \{\alpha_h = \mathbf{e}_h - \mathbf{e}_{h+1} \mid 1 \leq h \leq m+n\}$. Then \mathbf{U} has a natural grading over $\mathbb{Z}\Pi$:

$$\mathbf{U} = \bigoplus_{\nu \in \mathbb{Z}\Pi} \mathbf{U}_\nu \quad (4.1.3)$$

such that $K_i \in \mathbf{U}_0$, $E_h \in \mathbf{U}_{\alpha_i}$ and $F_h \in \mathbf{U}_{-\alpha_i}$. We will write $\text{gd}(x) = \nu$ if $x \in \mathbf{U}_\nu$, called the *graded degree* of x .

Consider the subspace $\mathfrak{A}(m|n)$ of the $\mathbb{Q}(\mathbf{v})$ -algebra

$$\mathcal{S}(m|n) := \prod_{r \geq 0} \mathcal{S}(m|n, r)$$

spanned by the linear independent set

$$\{A(\mathbf{j}) := \sum_{r \geq 0} A(\mathbf{j}, r) \mid A \in M(m|n)^\pm, \mathbf{j} \in \mathbb{Z}^{m+n}\}.$$

By [7, Ths. 9.1&9.4] (deduced from Proposition 3.2), $\mathfrak{A}(m|n)$ is a subalgebra isomorphic to $\mathbf{U}(\mathfrak{gl}_{m|n})$. Moreover, there is an algebra isomorphism given by

$$\eta : \mathbf{U}(\mathfrak{gl}_{m|n}) \longrightarrow \mathfrak{A}(m|n); E_h \mapsto E_{h, h+1}(\mathbf{0}), F_h \mapsto E_{h+1, h}(\mathbf{0}), K_i^{\pm 1} \mapsto O(\pm \mathbf{e}_i). \quad (4.1.4)$$

Let $U_{\mathcal{Z}}^+ = U_{\mathcal{Z}}^+(\mathfrak{gl}_{m|n})$ be the \mathcal{Z} -subalgebra of $\mathbf{U}(\mathfrak{gl}_{m|n})$ generated by all divided powers $E_h^{(l)} := \frac{E_h^l}{[l]_{\mathbf{v}_h}!}$, where $l \geq 1$ for all $h \neq m$. We have the following realisation of $U_{\mathcal{Z}}^+$.

Theorem 4.2. *The \mathcal{Z} -submodule $\mathfrak{A}_{\mathcal{Z}}^+$ spanned by*

$$\mathcal{B} = \{A(\mathbf{0}) \mid A \in M(m|n)^+\}$$

is a subalgebra of $\mathfrak{A}(m|n)$ which is isomorphic to $U_{\mathcal{Z}}^+$. In other words, we have $\eta(U_{\mathcal{Z}}^+) = \mathfrak{A}_{\mathcal{Z}}^+$.

Proof. The proof is somewhat standard; see e.g., [7, Th. 9.1]. Let \mathfrak{A}_1^+ be the \mathcal{Z} -subalgebra generated by $(lE_{h,h+1})(\mathbf{0})$ for all $l > 0$ and $h \in [1, m+n]$ ($l = 1$ if $h = m$). Then, by Lemmas 3.3 and 3.4, $\mathfrak{A}_1^+ \subseteq \mathfrak{A}_{\mathcal{Z}}^+$. Further, by Lemma 3.4, the triangular relation [7, (9.1.1)] can be taken over \mathcal{Z} (see (4.3.1) below). In particular, we can use this relation to prove that $\mathfrak{A}_{\mathcal{Z}}^+ \subseteq \mathfrak{A}_1^+$. \square

We will identify $\mathbf{U}(\mathfrak{gl}_{m|n})$ and $U_{\mathcal{Z}}^+$ with $\mathfrak{A}(m|n)$ and $\mathfrak{A}_{\mathcal{Z}}^+$, respectively, under η in the sequel.

We now take a closer look at the triangular relation mentioned in the proof. The order relation involved in the triangular relation is the following relation: for $A = (a_{i,j}), A' = (a'_{i,j}) \in M(m|n)$,

$$A' \preceq A \iff \begin{cases} (1) & \sum_{i \leq s, j \geq t} a'_{i,j} \leq \sum_{i \leq s, j \geq t} a_{i,j}, \quad \text{for all } s < t; \\ (2) & \sum_{i \geq s, j \leq t} a'_{i,j} \leq \sum_{i \geq s, j \leq t} a_{i,j}, \quad \text{for all } s > t. \end{cases} \quad (4.2.1)$$

Note that this definition is independent of the diagonal entries of a matrix. So \preceq is not a partial order on $M(m|n)$. However, its restriction to $M(m|n)^\pm$ is a partial order. In particular, we have posets $(M(m|n)^+, \preceq)$ and $(M(m|n)^-, \preceq)$.

Moreover, the following is taken from [1, Lem. 3.6(1)] (see also [6, Lem. 13.20, 13.21]): for $A, B \in M(m|n, r)$,

$$A \leq B \text{ (the Bruhat order)} \implies A \preceq B. \quad (4.2.2)$$

We may also introduce another partial order \preceq_{rc} on $M(m|n, r)$ defined by²

$$X \preceq_{\text{rc}} Y \iff \text{ro}(X) = \text{ro}(Y), \text{co}(X) = \text{co}(Y), \text{ and } X \preceq Y. \quad (4.2.3)$$

Remark 4.3. Since $X \leq Y \implies X \preceq_{\text{rc}} Y \implies X \preceq Y$, the canonical bases $\{\Xi_A \mid A \in M(m|n, r)\}$ defined in Corollary 2.4 can also be defined relative to the basis $\{[A]\}_A$, the bar involution and the order \preceq_{rc} .

For any $A = (a_{i,j}) \in M(m|n)^\pm$ and $\mathbf{j} \in \mathbb{Z}^{m+n}$, we have the following triangular relation in the $\mathbb{Q}(\mathbf{v})$ -algebra $\mathfrak{A}(m|n)$

$$\prod_{i,h,j}^{(\leq_2)} (a_{j,i} E_{h+1,h})(\mathbf{0}) \cdot \prod_{i,h,j}^{(\leq_1)} (a_{i,j} E_{h,h+1})(\mathbf{0}) = A(\mathbf{0}) + \sum_{\substack{B \in M(m|n)^\pm, \mathbf{j} \in \mathbb{Z}^{m+n} \\ B \prec A}} g_{B,A,\mathbf{j}} B(\mathbf{j}),$$

²This order relation is denoted by \sqsubseteq in [1].

where i, h, j satisfy $1 \leq i \leq h < j \leq m+n$ and the products follow the orders \leq_i which are defined as in [6, (13.7.1)]. In particular, by Lemma 3.4, a single product for $A \in M(m|n)^+$ can be simplified as

$$\mathbf{m}_A^+ := \prod_{1 \leq i \leq h < j \leq m+n}^{(\leq_1)} (a_{i,j} E_{h,h+1})(\mathbf{0}) = A(\mathbf{0}) + \sum_{B \in M(m|n)^+, B \prec A} g_{B,A} B(\mathbf{0}), \quad (4.3.1)$$

where $g_{B,A} \in \mathcal{Z}$. Then, applying the anti-involution τ in (4.1.1) yields

$$\mathbf{m}_{A^t}^- := \tau(\mathbf{m}_A^+) = \prod_{i,h,j}^{(\leq_1^{\text{op}})} (a_{i,j} E_{h+1,h})(\mathbf{0}). \quad (4.3.2)$$

Corollary 4.4. *The set $\{\mathbf{m}_A^+ \mid A \in M(m|n)^+\}$ (resp., $\{\mathbf{m}_A^- \mid A \in M(m|n)^-\}$) forms a \mathcal{Z} -basis, a monomial basis, for $U_{\mathcal{Z}}^+$ (resp., $U_{\mathcal{Z}}^-$).*

We end this section by showing that the basis $\mathcal{B} = \{A(\mathbf{0}) \mid A \in M(m|n)^+\}$ identifies a PBW basis for $U_{\mathcal{Z}}^+$.

For a root vector $E_{k,l}$ with $k < l$ and $p > 0$, if $E_{k,l}^p \neq 0$ define the usual divided powers $E_{k,l}^{(p)} = \frac{E_{k,l}^p}{[p]_{\mathbf{v}_k}!}$. If we order linearly the set

$$\mathcal{J}' = \{(i, j) \mid 1 \leq i < j \leq m+n\}$$

by setting, for $(i, j), (i', j') \in \mathcal{J}'$, $(i, j) <_3 (i', j')$ if and only if $j > j'$ or $j = j', i > i'$, and use the order to define, for any $A = (a_{i,j}) \in M(m|n)^+$, the product and its ‘transpose’

$$E_A = \prod_{(i,j) \in \mathcal{J}'}^{(\leq_3)} E_{i,j}^{(a_{i,j})} \quad \text{and} \quad F_{A^t} = \tau(E_A). \quad (4.4.1)$$

then the set $\{E_A \mid A \in M(m|n)^+\}$ (resp. $\{F_A \mid A \in M(m|n)^-\}$) forms a PBW basis of $U_{\mathcal{Z}}^+$ (resp., $U_{\mathcal{Z}}^-$). We now prove that this basis is nothing but the same basis given in Theorem 4.2.

For any $A \in M(m|n)$, set

$$\|A\| = \sum_{1 \leq i < j \leq m+n} \frac{(j-i)(j-i+1)}{2} (a_{i,j} + a_{j,i}).$$

Refer to [6, Lem. 13.21], for $A, B \in M(m|n)$,

$$B \prec A \implies \|B\| < \|A\| \quad (\text{and } B \preceq A \implies \|B\| \leq \|A\|).$$

Theorem 4.5. *For any $A \in M(m|n)^+$, we have $E_A = A(\mathbf{0})$. In other words, with the isomorphism η given in (4.1.4), we have $\eta(E_A) = A(\mathbf{0})$.*

Proof. Let $A = (a_{i,j})$. We apply induction on $\|A\|$ to prove the assertion. If $\|A\| = 1$, then A must be of the form $E_{i,i+1}$ for some $1 \leq i < m+n$. Thus, this case is clear from the definition of η . So $E_{i,i+1} = (E_{i,i+1})(\mathbf{0})$, as desired.

Assume now $\|A\| > 1$ and that, for any $B \in M(m|n)^+$ with $\|B\| < \|A\|$, $E_B = B(\mathbf{0})$. Consider the entries of A and choose $1 \leq h < l \leq m+n$ such that $a_{h,l} > 0$ and

$a_{i,j} = 0$ for all $j > l$ or $i > h$ whenever $j = l$. In other words, $\mathbf{E}_{h,l}^{(a_{h,l})}$ is the first factor in the product \mathbf{E}_A . Then, by the definition,

$$\mathbf{E}_A = \frac{1}{[a_{h,l}]_{\mathbf{v}_h}} \mathbf{E}_{h,l} \mathbf{E}_{A-E_{h,l}}.$$

Since $A - E_{h,l} \prec A$, $\|A - E_{h,l}\| < \|A\|$. By induction, we have $\mathbf{E}_{A-E_{h,l}} = (A - E_{h,l})(\mathbf{0})$ and also $\mathbf{E}_{h,l} = E_{h,l}(\mathbf{0})$. There are two cases to consider.

Case 1: $l = h + 1$. For this case, we directly use the multiplication formula given in Lemma 3.3. By the selection of indices h, l , all $a_{h+1,j} = 0 = a_{h,j}$ if $j > h + 1 = l$. Thus, by (3.3.1), $f_{A-E_{h,h+1}}(h, h+1) = a_{h,h+1} - 1$. So

$$\mathbf{E}_{h,h+1} \mathbf{E}_{A-E_{h,h+1}} = (E_{h,h+1})(\mathbf{0})(A - E_{h,h+1})(\mathbf{0}) = \mathbf{v}_h^{a_{h,h+1}-1} \overline{[a_{h,h+1}]_{\mathbf{v}_h}} A(\mathbf{0}).$$

But then $\mathbf{v}_h^{a_{h,h+1}-1} \overline{[a_{h,h+1}]_{\mathbf{v}_h}} = [a_{h,h+1}]_{\mathbf{v}_h}$. Hence, $\mathbf{E}_A = A(\mathbf{0})$, as desired.

Case 2: $l > h + 1$. In this case, write $\mathbf{E}_{h,l} = \mathbf{E}_{h,h+1} \mathbf{E}_{h+1,l} - \mathbf{v}_{h+1}^{-1} \mathbf{E}_{h+1,l} \mathbf{E}_{h,h+1}$. Since $A - E_{h,l} + E_{h+1,l} \prec A$, by induction,

$$\mathbf{E}_{h+1,l} \mathbf{E}_{A-E_{h,l}} = \mathbf{E}_{A-E_{h,l}+E_{h+1,l}} = (A - E_{h,l} + E_{h+1,l})(\mathbf{0}) \quad (4.5.1)$$

and, on the other hand,

$$\begin{aligned} \mathbf{E}_{h,h+1} \mathbf{E}_{A-E_{h,l}} &= (E_{h,h+1})(\mathbf{0})(A - E_{h,l})(\mathbf{0}) \\ &= (-1)^{\sigma_{A-E_{h,l}}(h+1)\delta_{h,m}} \mathbf{v}_h^{f_{A-E_{h,l}}(h,h+1)} \overline{[a_{h,h+1}+1]_{\mathbf{v}_h}} (A - E_{h,l} + E_{h,h+1})(\mathbf{0}) \\ &\quad + \sum_{a_{h+1,j} \geq 1} (-1)^{\sigma_{A-E_{h,l}}(j)\delta_{h,m}} \mathbf{v}_h^{f_{A-E_{h,l}}(h,j)} \overline{[a_{h,j}+1]_{\mathbf{v}_h}} (A - E_{h,l} + E_{h,j} - E_{h+1,j})(\mathbf{0}). \end{aligned} \quad (4.5.2)$$

Now multiplying (4.5.1) by $\mathbf{E}_{h,h+1}$ and applying Lemma 3.3 yields

$$\begin{aligned} \mathbf{E}_{h,h+1} \mathbf{E}_{h+1,l} \mathbf{E}_{A-E_{h,l}} &= (E_{h,h+1})(\mathbf{0})(A - E_{h,l} + E_{h+1,l})(\mathbf{0}) \\ &= (-1)^{\sigma(h+1)\delta_{h,m}} \mathbf{v}_h^{f(h,h+1)} \overline{[a_{h,h+1}+1]_{\mathbf{v}_h}} (A - E_{h,l} + E_{h+1,l} + E_{h,h+1})(\mathbf{0}) \\ &\quad + \sum_{a_{h+1,j} \geq 1, j \neq l} (-1)^{\sigma(j)\delta_{h,m}} \mathbf{v}_h^{f(h,j)} \overline{[a_{h,j}+1]_{\mathbf{v}_h}} (A - E_{h,l} + E_{h+1,l} + E_{h,j} - E_{h+1,j})(\mathbf{0}) \\ &\quad + \mathbf{v}_h^{a_{h,l}-1} \overline{[a_{h,l}]_{\mathbf{v}_h}} A(\mathbf{0}), \end{aligned} \quad (4.5.3)$$

where $\sigma(k) = \sigma_{A-E_{h,l}+E_{h+1,l}}(k) = \sigma_A(k) - \delta_{h,m}$ and $f(h,k) = f_{A-E_{h,l}+E_{h+1,l}}(h,k)$.

We now compute multiplying (4.5.2) by $\mathbf{E}_{h+1,l}$. Note that, since $a_{h+1,j} = 0$ for all $j \geq l$, the summation in (4.5.2) is taken over all j with $a_{h+1,j} \geq 1$ and $j < l$. But $j < l$ implies

$$A - E_{h,l} + E_{h,h+1} \prec A \text{ and } A - E_{h,l} + E_{h,j} - E_{h+1,j} \prec A.$$

Thus, by induction,

$$\begin{aligned} \mathbf{E}_{A-E_{h,l}+E_{h,h+1}} &= (A - E_{h,l} + E_{h,h+1})(\mathbf{0}) \\ \mathbf{E}_{A-E_{h,l}+E_{h,j}-E_{h+1,j}} &= (A - E_{h,l} + E_{h,j} - E_{h+1,j})(\mathbf{0}). \end{aligned}$$

Hence, by induction again,

$$\begin{aligned} \mathbf{E}_{h+1,l} \mathbf{E}_{A-E_{h,l}+E_{h,h+1}} &= \mathbf{E}_{A-E_{h,l}+E_{h,h+1}+E_{h+1,l}} = (A - E_{h,l} + E_{h,h+1} + E_{h+1,l})(\mathbf{0}) \\ \mathbf{E}_{h+1,l} \mathbf{E}_{A-E_{h,l}+E_{h,j}-E_{h+1,j}} &= \mathbf{E}_{A-E_{h,l}+E_{h,j}-E_{h+1,j}+E_{h+1,l}} \\ &= (A - E_{h,l} + E_{h,j} - E_{h+1,j} + E_{h+1,l})(\mathbf{0}), \end{aligned}$$

since $A - E_{h,l} + E_{h,h+1} + E_{h+1,l} \prec A$ and $A - E_{h,l} + E_{h,j} - E_{h+1,j} + E_{h+1,l} \prec A$.

Thus, for $\varepsilon = \delta_{h,m}$,

$$\begin{aligned} \mathbf{E}_{h+1,l} \mathbf{E}_{h,h+1} \mathbf{E}_{A-E_{h,l}} &= \mathbf{E}_{h+1,l} \cdot (\text{RHS of (4.5.2)}) \\ &= (-1)^{\varepsilon \sigma_{A-E_{h,l}}(h+1)} \mathbf{v}_h^{f_{A-E_{h,l}}(h,h+1)} \overline{[a_{h,h+1} + 1]}_{\mathbf{v}_h} (A - E_{h,l} + E_{h,h+1} + E_{h+1,l})(\mathbf{0}) \\ &+ \sum_{a_{h+1,j} \geq 1} (-1)^{\varepsilon(\sigma_{A-E_{h,l}}(j))} \mathbf{v}_h^{f_{A-E_{h,l}}(h,j)} \overline{[a_{h,j} + 1]}_{\mathbf{v}_h} (A - E_{h,l} + E_{h,j} - E_{h+1,j} + E_{h+1,l})(\mathbf{0}). \end{aligned} \quad (4.5.4)$$

Finally, since

$$f(h, j) = f_{A-E_{h,l}}(h, j) - (-1)^\varepsilon,$$

and $\sigma(j) - \varepsilon = \sigma_{A-E_{h,l}}(j)$ for $h = m$, Combining (4.5.3) and (4.5.4) gives

$$\mathbf{E}_{h,l} \mathbf{E}_{A-E_{h,l}} = \mathbf{E}_{h,h+1} \mathbf{E}_{h+1,l} \mathbf{E}_{A-E_{h,l}} - \mathbf{v}_{h+1}^{-1} \mathbf{E}_{h+1,l} \mathbf{E}_{h,h+1} \mathbf{E}_{A-E_{h,l}} = [a_{h,l}]_{\mathbf{v}_h} A(\mathbf{0}),$$

proving $\mathbf{E}_A = A(\mathbf{0})$. \square

5. CANONICAL BASIS FOR $U_{\mathcal{Z}}^\pm(\mathfrak{gl}_{m|n})$

We are now ready to introduce the canonical basis for $U_{\mathcal{Z}}^\pm$ via the PBW basis described in (4.4.1) and the partial order \preceq used in the triangular relation (4.3.1). We need another ingredient—a bar involution.

By Definition 4.1, we may define the bar involution

$$\bar{\cdot} : \mathbf{U}(\mathfrak{gl}_{m|n}) \rightarrow \mathbf{U}(\mathfrak{gl}_{m|n}) \text{ with } \bar{\mathbf{v}} = \mathbf{v}^{-1}, \bar{\mathbf{E}}_h = \mathbf{E}_h, \bar{\mathbf{F}}_h = \mathbf{F}_h, \bar{\mathbf{K}}_i^\pm = \mathbf{K}_i^\mp. \quad (5.0.5)$$

Remark 5.1. (1) If we denote \flat to be the involution on the direct product $\mathcal{S}(m|n) = \prod_{r \geq 0} \mathcal{S}(m|n, r)$ defined by baring on every component (see (2.2.1)), then the restriction of \flat to $\mathfrak{A}(m|n) = \mathbf{U}(\mathfrak{gl}_{m|n})$ coincides with the bar involution on $\mathbf{U}(\mathfrak{gl}_{m|n})$. This can be seen as follows.

If $A = \text{diag}(\lambda)$ or $\text{diag}(\lambda) + E_{h,h+1}$, then A is minimal under the Bruhat ordering. Thus, (2.2.1) implies $\overline{[A]} = [A]$. Since $E_{h,h+1}(\mathbf{0}, r) = \sum_{\lambda \in \Lambda(m|n, r-1)} [E_{h,h+1} + \text{diag}(\lambda)]$ and $O(\mathbf{e}_i, r) = \sum_{\lambda \in \Lambda(m|n, r)} \mathbf{v}_i^{\lambda_i} [\text{diag}(\lambda)]$, it follows that $\overline{E_{h,h+1}(\mathbf{0}, r)} = E_{h,h+1}(\mathbf{0}, r)$ and $\overline{O(\mathbf{e}_i, r)} = O(-\mathbf{e}_i, r)$. Similarly, $\overline{E_{h+1,h}(\mathbf{0}, r)} = E_{h+1,h}(\mathbf{0}, r)$. Hence, $\flat(E_{h,h+1}(\mathbf{0})) = E_{h,h+1}(\mathbf{0})$, $\flat(O(\mathbf{e}_i)) = O(-\mathbf{e}_i)$, and $\flat(E_{h+1,h}(\mathbf{0})) = E_{h+1,h}(\mathbf{0})$. This is the same bar involution as defined in (5.0.5).

(2) Matrix transposing induces $\mathbb{Q}(\mathbf{v})$ -algebra anti-involutions

$$\tau_r : \mathcal{S}(m|n, r) \longrightarrow \mathcal{S}(m|n, r), \quad \xi_A \longmapsto \xi_{A^t},$$

which induce anti-involution τ on $\mathcal{S}(m|n)$ and, hence, on $\mathfrak{A}(m|n) = \mathbf{U}(\mathfrak{gl}_{m|n})$. This is the same τ as defined in (4.1.1). Hence, $\tau(A(\mathbf{0})) = A^t(\mathbf{0})$ for all $A \in M(m|n)^+$. In particular, by (4.4.1), we have $F_A = A(\mathbf{0})$ for all $A \in M(m|n)^-$.

(3) Let $\iota = \flat \circ \tau = \tau \circ \flat$. Then ι is a ring anti-involution

$$\iota : \mathbf{U} \longrightarrow \mathbf{U}, \quad E_h \longmapsto F_h, F_h \longmapsto E_h, K_i^{\pm 1} \longmapsto K_i^{\mp 1}, \mathbf{v}^{\pm 1} \longmapsto \mathbf{v}^{\mp 1}. \quad (5.1.1)$$

Note that $\iota(E_{a,b}) = E_{b,a}$ for all $a \neq b$ (cf. the remark after (4.1.2)).

Theorem 5.2. *The basis $\{A(\mathbf{0})\}_{A \in M(m|n)^+}$, the bar involution, and the partial order \preceq define uniquely the canonical $\mathcal{C}^+ = \{\mathbf{C}_A \mid A \in M(m|n)^+\}$ for $U_{\mathcal{Z}}^+$. In other words, the elements \mathbf{C}_A are uniquely defined by the conditions $\bar{\mathbf{C}}_A = \mathbf{C}_A$ and*

$$\mathbf{C}_A - A(\mathbf{0}) \in \sum_{B \in M(m|n)^+, B \prec A} \mathbf{v}^{-1} \mathbb{Z}[\mathbf{v}^{-1}] B(\mathbf{0}).$$

Applying the anti-involution τ yields the canonical basis

$$\mathcal{C}^- = \{\mathbf{C}_A = \tau(\mathbf{C}_{A^t}) \mid A \in M(m|n)^-\} \quad (5.2.1)$$

of $U_{\mathcal{Z}}^-$ which can be defined similarly relative to $\{A(\mathbf{0})\}_{A \in M(m|n)^-}$ etc.

Proof. By (4.3.1), we may write the basis $\{A(\mathbf{0})\}_{A \in M(m|n)^+}$ in terms of the monomial basis:

$$A(\mathbf{0}) = \mathbf{m}_A^+ + \sum_{B \in M(m|n)^+, B \prec A} h_{B,A} \mathbf{m}_B^+,$$

where $h_{B,A} \in \mathcal{Z}$. Applying the bar involution and (4.3.1) yields

$$\begin{aligned} \overline{A(\mathbf{0})} &= \mathbf{m}_A^+ + \sum_{B \in M(m|n)^+, B \prec A} \overline{h_{B,A}} \mathbf{m}_B^+ \\ &= A(\mathbf{0}) + \sum_{A' \in M(m|n)^+, A' \prec A} f_{A',A} A'(\mathbf{0}) \quad (f_{A',A} \in \mathcal{Z}). \end{aligned}$$

Thus, a standard construction (see, e.g., [6, §0.5]) shows that there exist polynomials $p_{B,A}$ with $p_{A,A} = 1$ and $p_{B,A} \in \mathbf{v}^{-1} \mathbb{Z}[\mathbf{v}^{-1}]$ if $B \prec A$ such that the elements

$$\mathbf{C}_A = A(\mathbf{0}) + \sum_{B \in M(m|n)^+, B \prec A} p_{B,A} B(\mathbf{0}) \quad (A \in M(m|n)^+) \quad (5.2.2)$$

form the required basis. For the last assertion, see (4.3.2) and Remark 5.1(2). \square

Recall from [7, Cor.6.4] that there are $\mathbb{Q}(\mathbf{v})$ -superalgebra epimorphisms

$$\eta_r : \mathbf{U}(\mathfrak{gl}_{m|n}) \longrightarrow \mathcal{S}(m|n, r) \quad (5.2.3)$$

sending E_h, F_h and $K^{\pm 1}$ to $E_{h,h+1}(\mathbf{0}, r), E_{h+1,h}(\mathbf{0}, r)$ and $O(\pm e_i, r)$, respectively. Note that it was these epimorphisms that induce the isomorphism η in (4.1.4). Note also that the epimorphism η_r is compatible with the bar involutions by the remark above.

Let $\mathcal{S}^-(m|n, r) = \eta_r(U_{\mathcal{Z}}^-)$ and $\mathcal{S}^+(m|n, r) = \eta_r(U_{\mathcal{Z}}^+)$. The subalgebra $\mathcal{S}^-(m|n, r)$ (resp. $\mathcal{S}^+(m|n, r)$) has a basis

$$\mathcal{B}_r^- = \{A(\mathbf{0}, r) \mid A \in M(m|n)_{\preceq_r}^-\} \quad (\text{resp. } \mathcal{B}_r^+ = \{A(\mathbf{0}, r) \mid A \in M(m|n)_{\preceq_r}^+\}),$$

where $M(m|n)_{\leq r}^* = \{A \in M(m|n)^*, |A| \leq r\}$ for $* = +, -$.

Corollary 5.3. *For any $r > 0$, let $\mathbf{c}_A = \eta_r(\mathbf{C}_A)$ for all $A \in M(m|n)_{\leq r}^+$. Then $\{\mathbf{c}_A\}_{A \in M(m|n)_{\leq r}^+}$ forms a basis for $\mathcal{S}^+(m|n, r)$ which satisfies the following properties:*

$$\overline{\mathbf{c}_A} = \mathbf{c}_A \text{ and } \mathbf{c}_A - A(\mathbf{0}, r) \in \sum_{B \prec A} \mathbf{v}^{-1} \mathbb{Z}[\mathbf{v}^{-1}] B(\mathbf{0}, r).$$

In other words, this is the canonical basis relative to \mathcal{B}_r^+ and the restrictions of the bar involution (2.2.1) and \preceq . Moreover, we have

$$\eta_r(\mathbf{C}_A) = \begin{cases} \mathbf{c}_A, & \text{if } A \in M(m|n)_{\leq r}^+, \\ 0, & \text{otherwise.} \end{cases} \quad (5.3.1)$$

A similar result holds for $\mathcal{S}^-(m|n, r)$.

Proof. Applying η_r to (4.3.1) yields a triangular relation in $\mathcal{S}^+(m|n, r)$ between the monomial basis and \mathcal{B}_r^+ . So the canonical basis, defined by the basis \mathcal{B}_r^+ , the bar involution and the order \preceq , exists. By Remark 5.1 and (5.2.2), \mathbf{c}_A does satisfy the described conditions. Hence, it is the required canonical basis. \square

We now make a comparison between the canonical bases \mathcal{C}^- for $U_{\mathcal{Z}}^-$ and \mathcal{C}_r for the quantum Schur superalgebra given in Corollary 2.4. Note that \mathcal{C}_r can be defined by using the order \preceq_{rc} (see Remark 4.3).

Let $U_{\mathcal{Z}}^0$ be the \mathcal{Z} subalgebra of $\mathbf{U}(\mathfrak{gl}_{m|n})$ generated by K_i and $\begin{bmatrix} K_i; 0 \\ t \end{bmatrix}$ for $1 \leq i \leq m+n$ and $t \geq 1$, where³

$$\begin{bmatrix} K_i; 0 \\ t \end{bmatrix} = \prod_{a=1}^t \frac{K_i \mathbf{v}_i^{-a+1} - K_i^{-1} \mathbf{v}_i^{a-1}}{\mathbf{v}_i^a - \mathbf{v}_i^{-a}}.$$

Since, for any $\lambda \in \Lambda(m|n, r)$, $\eta_r(\prod_{i=1}^{m+n} \begin{bmatrix} K_i; 0 \\ \lambda_i \end{bmatrix}) = [\text{diag}(\lambda)]$, the \mathcal{Z} -subalgebra $\mathcal{S}^{\geq 0}(m|n, r) := \eta_r(U_{\mathcal{Z}}^0 U_{\mathcal{Z}}^+)$ is spanned by $\{[\text{diag}(\mu)] A(\mathbf{0}, r) \mid A \in M(m|n, r)^+, \mu \in \Lambda(m|n, r)\}$. This is called a Borel subsuperalgebra.

For $A \in M(m|n)$, define the “hook sums”

$$\mathfrak{h}_i(A) = a_{i,i} + \sum_{i < j} (a_{i,j} + a_{j,i}) \text{ and } \mathfrak{h}(A) = (\mathfrak{h}_1(A), \dots, \mathfrak{h}_{m+n}(A)).$$

If we write $A = A^- + A^0 + A^+$, where $A^{\pm} \in M(m|n)^{\pm}$ and A^0 is diagonal, then

$$\mathfrak{h}(A) = (a_{1,1}, \dots, a_{n,n}) + \text{co}(A^-) + \text{ro}(A^+).$$

Set $\mathfrak{h}(A) \leq \lambda$ if and only if $\mathfrak{h}_i(A) \leq \lambda_i$ for every i .

For $A \in M(m|n)^-$ and $\lambda \in \Lambda(m|n, r)$ with $\mathfrak{h}(A) = \text{co}(A) \leq \lambda$, let

$$A_{\lambda} := A + \text{diag}(\lambda - \mathfrak{h}(A)). \quad (5.3.2)$$

It is clear that $[A_{\lambda}] = A(\mathbf{0}, r)[\text{diag}(\lambda)]$ and the set

$$\{[A_{\lambda}] \mid A \in M(m|n, r)^-, \lambda \in \Lambda(m|n, r), \mathfrak{h}(A) \leq \lambda\}$$

³We have corrected typos in line 3 from [7, Th. 8.4].

forms a basis for $\mathcal{S}^{\leq 0}(m|n, r)$.

We have the following theorem (cf. [9, Th. 8.3]).

Theorem 5.4. *For any $A \in M(m|n)^-$, if $|A| \leq r$, then $\mathbf{c}_A = \sum_{\mathbf{h}(A) \leq \lambda} (-1)^{\bar{A}_\lambda} \Xi_{A_\lambda}$. In other words, the image \mathbf{c}_A of the canonical basis element $\mathbf{C}_A \in U_{\mathcal{Z}}^-$ under η_r is either zero or a sum of the canonical basis elements $\Xi_{A_\lambda} = \mathbf{c}_A[\text{diag}(\lambda)] \in \mathcal{S}(m|n, r)$ ($\mathbf{h}(A) \leq \lambda$). Moreover,*

$$\mathcal{C}_r^{\leq 0} = \{\Xi_{A_\lambda} \mid A \in M(m|n, \leq r)^-, \lambda \in \Lambda(m|n, r), \sigma_i(A) \leq \lambda_i\} = \mathcal{C}_r \cap \mathcal{S}^{\leq 0}(m|n, r)$$

forms the canonical basis for the Borel subsuperalgebra $\mathcal{S}^{\leq 0}(m|n, r)$.

A similar result holds for $\mathcal{S}^{\geq 0}(m|n, r)$

Proof. By Remark 5.1 and Corollary 5.3, we have $\overline{\mathbf{c}_A} = \mathbf{c}_A$ and $\overline{[\text{diag}(\lambda)]} = [\text{diag}(\lambda)]$. Hence, $\overline{\mathbf{c}_A[\text{diag}(\lambda)]} = \mathbf{c}_A[\text{diag}(\lambda)]$.

Since, by definition, $\mathbf{c}_A = A(\mathbf{0}, r) + \sum_{B \prec A} p_{B,A} B(\mathbf{0}, r)$, where $p_{B,A} \in \mathbf{v}^{-1}\mathbb{Z}[\mathbf{v}^{-1}]$, it follows that

$$\begin{aligned} \mathbf{c}_A[\text{diag}(\lambda)] &= A(\mathbf{0}, r)[\text{diag}(\lambda)] + \sum_{B \prec A} p_{B,A} B(\mathbf{0}, r)[\text{diag}(\lambda)] \\ &= (-1)^{\bar{A}_\lambda} [A_\lambda] + \sum_{B \prec A} p_{B,A} (-1)^{\bar{B}_\lambda} [B_\lambda]. \end{aligned}$$

By definition, $B \prec A$ implies $B_\lambda \prec A_\lambda$. Also, $\lambda = \text{co}(A_\lambda)$. Thus, if $\mu = \text{ro}(A_\lambda)$, then $\mathbf{c}_A[\text{diag}(\lambda)] = [\text{diag}(\mu)]\mathbf{c}_A$. By Corollary 5.3, we obtain

$$(-1)^{\bar{A}_\lambda} \mathbf{c}_A[\text{diag}(\lambda)] = [A_\lambda] + \sum_{B_\lambda \prec_{\text{rc}} A_\lambda} (-1)^{\bar{A}_\lambda + \bar{B}_\lambda} p_{B,A} [B_\lambda],$$

where \preceq_{rc} is the partial order relation on $M(m|n, r)$ defined in (4.2.3). Now, by Remark 4.3 and the uniqueness of canonical basis, we must have $(-1)^{\bar{A}_\lambda} \mathbf{c}_A[\text{diag}(\lambda)] = \Xi_{A_\lambda}$. \square

We end this section with description of a PBW type basis for $\mathcal{S}(m|n, r)$.

Corollary 5.5. *Maintain the notation in (5.3.2). For $A \in M(m|n)^\pm$ and $\lambda \in \Lambda(m|n, r)$, we have*

$$A^-(\mathbf{0}, r)[\text{diag}(\lambda)]A^+(\mathbf{0}, r) = \varepsilon_{\lambda, \mathbf{h}(A)} (-1)^{\bar{A}_\lambda} [A_\lambda] + (\text{lower terms w.r.t. } \preceq_{\text{rc}}),$$

where $\varepsilon_{\lambda, \mathbf{h}(A)} = 1$ if $\lambda \geq \mathbf{h}(A)$ and 0 otherwise. In particular, the set

$$\{A^-(\mathbf{0}, r)[\text{diag}(\lambda)]A^+(\mathbf{0}, r) \mid A \in M(m|n)^\pm, \lambda \in \Lambda(m|n, r), \lambda \geq \mathbf{h}(A)\}$$

forms a \mathcal{Z} -basis for $\mathcal{S}(m|n, r)$.

Proof. For $A \in M(m|n)^\pm$, write $A = A^+ + A^-$ where $A^+ \in M(m|n)^+$ and $A^- \in M(m|n)^-$. Then $\mathbf{h}(A) = \text{ro}(A^+) + \text{co}(A^-)$. Since

$$A^-(\mathbf{0}, r)[\text{diag}(\lambda)]A^+(\mathbf{0}, r) = A^-(\mathbf{0}, r)A^+(\mathbf{0}, r)[\text{diag}(\lambda + \text{co}(A^+) - \text{ro}(A^+))]$$

and, by [7, (8.1.1)] and the argument after [7, (8.1.2)] (cf. [7, Th. 7.1], [6, Th. 13.44]),

$$A^-(\mathbf{0}, r)A^+(\mathbf{0}, r) = A(\mathbf{0}, r) + (\text{lower terms w.r.t. } \preceq_{\text{rc}}),$$

the first assertion follows the fact that $A(\mathbf{0}, r)[\text{diag}(\lambda + \text{co}(A^+) - \text{ro}(A^+))] \neq 0$ implies $\text{co}(A) + \mu = \lambda + \text{co}(A^+) - \text{ro}(A^+)$ for some $\mu \in \Lambda(m|n, r - |A|)$ (so $\mu = \lambda - \mathbf{h}(A)$). The last assertion is clear since, if $\lambda \geq \mathbf{h}(A)$, then $A^-(\mathbf{0}, r)[\text{diag}(\lambda)]A^+(\mathbf{0}, r)$ has the leading term $(-1)^{\overline{A\lambda}}[A_\lambda]$. \square

6. EXAMPLES: CANONICAL BASES FOR $U_{\mathcal{Z}}^+(\mathfrak{gl}_{2|1})$ AND $U_{\mathcal{Z}}^+(\mathfrak{gl}_{2|2})$

Recall from §5 that the relation (4.3.1) was used to show the existence of canonical bases. In this section, we will see how this relation is used to compute the canonical basis element C_A .

Let $A \in M(m|n)^+$. We use the order \leq_1 to write down the monomial \mathbf{m}_A^+ , i.e., the left hand side of (4.3.1). Then apply the formula in Lemma 3.4 to compute the right hand side of (4.3.1):

$$\mathbf{m}_A^+ = A(\mathbf{0}) + \sum_{B \in M(m|n)^+, B \prec A} g_{B,A} B(\mathbf{0}) \quad (g_{B,A} \in \mathcal{Z}).$$

If all $g_{B,A} \in \mathcal{Z}^- := \mathbf{v}^{-1}\mathbb{Z}[\mathbf{v}^{-1}]$, then by Theorem 5.2, $C_A = \mathbf{m}_A^+$. Suppose now some $g_{B,A} \notin \mathcal{Z}^-$. Partition the poset ideal

$$\mathcal{I}_{\prec A} = \{B \in M(m|n)^+ \mid B \prec A\} = \mathcal{I}_{\prec A}^1 \cup \mathcal{I}_{\prec A}^2 \cup \cdots \cup \mathcal{I}_{\prec A}^t,$$

into subsets $\mathcal{I}_{\prec A}^i$ which consists of the maximal elements of $\mathcal{I}_{\prec A} \setminus \bigcup_{j=1}^{i-1} \mathcal{I}_{\prec A}^j$ for all $1 \leq i \leq t$. In particular, $\mathcal{I}_{\prec A}^1$ consists of all maximal elements of $\mathcal{I}_{\prec A}$. Choose B so that $g_{B,A} \notin \mathcal{Z}^-$ and $B \in \mathcal{I}_{\prec A}^a$ with a minimal. Thus, $g_{B',A} \in \mathcal{Z}^-$ if $B \prec B'$ or $B' \in \mathcal{I}_{\prec A}^i$ for some $i < a$. Since every polynomial $g \in \mathcal{Z}$ can be written as $g' + g''$ with $g' \in \mathcal{Y} := \{h + \bar{h} \mid h \in \mathbb{Z}[\mathbf{v}]\}$ and $g'' \in \mathcal{Z}^-$, there exist $f_{B,A} \in \mathcal{Z}$ and $g'_{B,A} \in \mathcal{Y}$ such that

$$\mathbf{m}_A^+ - \sum_{B \in \mathcal{I}_{\prec A}^a} g'_{B,A} \mathbf{m}_B^+ - \left(A(\mathbf{0}) + \sum_{B \in \mathcal{I}_{\prec A}^i, i > a} f_{B,A} B(\mathbf{0}) \right) \in \sum_{B \prec A} \mathcal{Z}^- B(\mathbf{0}).$$

By a similar argument with $f_{B,A}$ and continuing if necessary, we can eventually find an $\mathbf{m} \in \sum_{B \prec A} \mathcal{Y} \mathbf{m}_B^+$ such that

$$\mathbf{m}_A^+ + \mathbf{m} - A(\mathbf{0}) \in \sum_{B \prec A} \mathcal{Z}^- B(\mathbf{0}).$$

Since $\overline{\mathbf{m}_A^+ + \mathbf{m}} = \mathbf{m}_A^+ + \mathbf{m}$, we must have $C_A = \mathbf{m}_A^+ + \mathbf{m}$ by Theorem 5.2. We have proved the following.

Lemma 6.1. *For $A \in M(m|n)^+$, there exists $\mathbf{m} \in \sum_{B \prec A} \mathcal{Y} \mathbf{m}_B^+$ such that the canonical basis element $C_A = \mathbf{m}_A^+ + \mathbf{m}$.*

This algorithm has been used before (see, e.g., [3]). We now use the algorithm to compute some small rank examples.

Example 6.2. The canonical basis of $U_{\mathcal{Z}}^+(\mathfrak{gl}_{2|1})$ has been found in [13] and [5, 8.1]. We now follow the algorithm via our multiplication formulas to show that it consists of

$$\begin{aligned} E_1^{(a)} &= (aE_{1,2})(\mathbf{0}), \\ E_2 E_1^{(a)} &= (E_{2,3} + aE_{1,2})(\mathbf{0}), \\ E_1 E_2 E_1^{(a)} - [a]E_2 E_1^{(a+1)} &= (aE_{1,2} + E_{1,3})(\mathbf{0}) + \mathbf{v}^{-(a+1)}((a+1)E_{1,2} + E_{2,3})(\mathbf{0}), \\ E_2 E_1 E_2 E_1^{(a)} &= (aE_{1,2} + E_{1,3} + E_{2,3})(\mathbf{0}). \end{aligned} \tag{6.2.1}$$

We observe the following. From [15, Example 3.4], $U(\mathfrak{gl}_3)$ has the canonical basis consisting of tight monomials

$$\{E_1^{(b)} E_2^{(b+c)} E_1^{(a)} \mid c \geq a\} \cup \{E_2^{(c)} E_1^{(a+b)} E_2^{(b)} \mid c < a\}.$$

If we regard E_2 as an odd generator and only consider the power $E_2^{(a)}$ with $a = 0, 1$, we obtain the following elements from the classical canonical basis above:

$$E_1^{(a)}, \quad E_1^{(a+1)} E_2, \quad E_2 E_1^{(a)}, \quad E_2 E_1^{(a+1)} E_2.$$

Then we claim that they coincide with the canonical basis above. Indeed, by the multiplication formulas given in Lemma 3.4, we have

$$\begin{aligned} E_1^{(a+1)} E_2 &= (aE_{1,2} + E_{1,3})(\mathbf{0}) + \mathbf{v}^{-(a+1)}((a+1)E_{1,2} + E_{2,3})(\mathbf{0}), \\ E_2 E_1^{(a+1)} E_2 &= (aE_{1,2} + E_{1,3} + E_{2,3})(\mathbf{0}), \end{aligned}$$

which are the third and fourth elements in (6.2.1). Hence, the canonical basis for $U_{\mathcal{Z}}^+(\mathfrak{gl}_{2|1})$ consists of tight monomials.

We now compute the canonical basis of $U_{\mathcal{Z}}^+(\mathfrak{gl}_{2|2})$. We will use the following abbreviation for a 4×4 matrix in $M(2|2)^+$:

$$A = \begin{bmatrix} a & b & d \\ & c & e \\ & & f \end{bmatrix} := \begin{pmatrix} 0 & a & b & d \\ 0 & 0 & c & e \\ 0 & 0 & 0 & f \\ 0 & 0 & 0 & 0 \end{pmatrix} \in M(2|2)^+$$

where $a, f \in \mathbb{Z}_{\geq 0}, b, c, d, e \in \{0, 1\}$.

Example 6.3. The canonical basis of $U_{\mathcal{Z}}^+(\mathfrak{gl}_{2|2})$ is listed in the following 18 cases. Each case is displayed in the form: $A, \mathbf{m}_A^+ + \mathbf{m} = \mathbf{C}_A$ as in Lemma 6.1 where \mathbf{m} is a \mathcal{Y} -linear combinations of monomial basis.

- (0) $A = \begin{bmatrix} a & 0 & 0 \\ 0 & 0 & f \end{bmatrix}$, $E_3^{(f)} E_1^{(a)} = A(\mathbf{0})$. This is the only even case.
- (1) $A = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 0 \\ & & f \end{bmatrix}$, $E_3^{(f)} E_2 E_1^{(a)} = A(\mathbf{0})$.
- (2) $A = \begin{bmatrix} a & 1 & 0 \\ 0 & 0 & 0 \\ & & f \end{bmatrix}$, $E_3^{(f)} E_1 E_2 E_1^{(a)} - [a]E_3^{(f)} E_2 E_1^{(a+1)} = A(\mathbf{0}) + \mathbf{v}^{-a-1} \begin{bmatrix} a+1 & 0 & 0 \\ 1 & 0 & f \end{bmatrix}(\mathbf{0})$.

$$(3) \quad A = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 1 \\ f & & \end{bmatrix}, \quad E_3^{(f)} E_2 E_3 E_1^{(a)} - [f+2] E_3^{(f+1)} E_2 E_1^{(a)} = A(\mathbf{0}) - \mathbf{v}^{-f-1} \begin{bmatrix} a & 0 & 0 \\ 1 & 0 & 0 \\ f+1 & & \end{bmatrix} (\mathbf{0}).$$

$$(4a) \quad A = \begin{bmatrix} a & 0 & 1 \\ 0 & 0 & 1 \\ f & & \end{bmatrix} \text{ with } a \leq f,$$

$$\begin{aligned} & E_3^{(f)} E_1 E_2 E_3 E_1^{(a)} - [a] E_3^{(f)} E_2 E_3 E_1^{(a+1)} - [f+2] E_3^{(f)} E_1 E_2 E_1^{(a+1)} \\ & + (2[a][f+2] + [f-a+1] - [a+1][f+1]) E_3^{(f+1)} E_2 E_1^{(a+1)} \\ & = A(\mathbf{0}) + \mathbf{v}^{-a-1} \begin{bmatrix} a+1 & 0 & 0 \\ 0 & 1 & 1 \\ f & & \end{bmatrix} (\mathbf{0}) - \mathbf{v}^{-f-1} \begin{bmatrix} a+1 & 1 & 0 \\ 0 & 0 & 0 \\ f+1 & & \end{bmatrix} (\mathbf{0}) - \mathbf{v}^{-f-a-2} \begin{bmatrix} a+1 & 0 & 0 \\ 1 & 0 & 0 \\ f+1 & & \end{bmatrix} (\mathbf{0}) \end{aligned}$$

$$(4b) \quad A = \begin{bmatrix} a & 0 & 1 \\ 0 & 0 & 1 \\ f & & \end{bmatrix} \text{ with } f = a-1,$$

$$\begin{aligned} & E_3^{(f)} E_1 E_2 E_3 E_1^{(a)} - [a] E_3^{(f)} E_2 E_3 E_1^{(a+1)} - [f+2] E_3^{(f)} E_1 E_2 E_1^{(a+1)} \\ & + (2[a][f+2] - [a+1][f+1]) E_3^{(f+1)} E_2 E_1^{(a+1)} \\ & = A(\mathbf{0}) + \mathbf{v}^{-a-1} \begin{bmatrix} a+1 & 0 & 0 \\ 0 & 1 & 1 \\ f & & \end{bmatrix} (\mathbf{0}) - \mathbf{v}^{-f-1} \begin{bmatrix} a+1 & 1 & 0 \\ 0 & 0 & 0 \\ f+1 & & \end{bmatrix} (\mathbf{0}) + (*) \begin{bmatrix} a+1 & 0 & 0 \\ 1 & 0 & 0 \\ f+1 & & \end{bmatrix} (\mathbf{0}) \end{aligned}$$

$$\text{where } * = \mathbf{v}^{-f-1}[a] - \mathbf{v}^{-a-1}[f+2].$$

$$(4c) \quad A = \begin{bmatrix} a & 0 & 1 \\ 0 & 0 & 1 \\ f & & \end{bmatrix} \text{ with } f \leq a-2,$$

$$\begin{aligned} & E_3^{(f)} E_1 E_2 E_3 E_1^{(a)} - [a] E_3^{(f)} E_2 E_3 E_1^{(a+1)} - [f+2] E_3^{(f)} E_1 E_2 E_1^{(a+1)} \\ & + (2[a][f+2] - [a-f-1] - [a+1][f+1]) E_3^{(f+1)} E_2 E_1^{(a+1)} \\ & = A(\mathbf{0}) + \mathbf{v}^{-a-1} \begin{bmatrix} a+1 & 0 & 0 \\ 0 & 1 & 1 \\ f & & \end{bmatrix} (\mathbf{0}) - \mathbf{v}^{-f-1} \begin{bmatrix} a+1 & 1 & 0 \\ 0 & 0 & 0 \\ f+1 & & \end{bmatrix} (\mathbf{0}) - \mathbf{v}^{-f-a-2} \begin{bmatrix} a+1 & 0 & 0 \\ 1 & 0 & 0 \\ f+1 & & \end{bmatrix} (\mathbf{0}). \end{aligned}$$

$$(5) \quad A = \begin{bmatrix} a & 1 & 0 \\ 0 & 1 & 0 \\ f & & \end{bmatrix}, \quad E_3^{(f)} E_2 E_1 E_2 E_1^{(a)} = A(\mathbf{0}).$$

$$(6) \quad A = \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 1 \\ f & & \end{bmatrix}, \quad E_3^{(f)} E_2 E_3 E_2 E_1^{(a)} = A(\mathbf{0}).$$

$$\begin{aligned} (7) \quad A &= \begin{bmatrix} a & 1 & 0 \\ 0 & 1 & 1 \\ f & & \end{bmatrix}, \quad E_3^{(f)} E_2 E_3 E_1 E_2 E_1^{(a)} - [a] E_3^{(f)} E_2 E_3 E_2 E_1^{(a+1)} - [f+2] E_3^{(f+1)} E_2 E_1 E_2 E_1^{(a)} \\ &= A(\mathbf{0}) + \mathbf{v}^{-a-1} \begin{bmatrix} a+1 & 0 & 0 \\ 1 & 1 & 1 \\ f & & \end{bmatrix} (\mathbf{0}) - \mathbf{v}^{-f-1} \begin{bmatrix} a & 1 & 0 \\ 1 & 0 & 0 \\ f+1 & & \end{bmatrix} (\mathbf{0}) \end{aligned}$$

$$(8a) \quad A = \begin{bmatrix} a & 0 & 1 \\ 1 & 0 & 1 \\ f & & \end{bmatrix} (a=0), \quad E_3^{(f)} E_1 E_2 E_3 E_2 = A(\mathbf{0}) + \mathbf{v}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ f+1 & & \end{bmatrix} (\mathbf{0}) + \mathbf{v}^{-2} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ f & & \end{bmatrix} (\mathbf{0}).$$

$$\begin{aligned} (8b) \quad A &= \begin{bmatrix} a & 0 & 1 \\ 0 & 1 & 0 \\ f & & \end{bmatrix} (a>0), \quad E_3^{(f)} E_1 E_2 E_3 E_2 E_1^{(a)} - [a-1] E_3^{(f)} E_2 E_3 E_2 E_1^{(a+1)} \\ &= A(\mathbf{0}) + \mathbf{v}^{-1} \begin{bmatrix} a & 0 & 0 \\ 0 & 1 & 1 \\ f+1 & & \end{bmatrix} (\mathbf{0}) + (\mathbf{v}^{-a} + \mathbf{v}^{-a-2}) \begin{bmatrix} a+1 & 0 & 0 \\ 1 & 1 & 1 \\ f & & \end{bmatrix} (\mathbf{0}). \end{aligned}$$

$$(9) \quad A = \begin{bmatrix} a & 1 & 1 \\ 0 & 0 & 0 \\ f & & \end{bmatrix},$$

$$\begin{aligned} & E_3^{(f)} E_1 E_2 E_3 E_1 E_2 E_1^{(a)} - [a] E_3^{(f)} E_1 E_2 E_3 E_2 E_1^{(a+1)} \\ & - [a+1] E_3^{(f)} E_2 E_3 E_1 E_2 E_1^{(a+1)} + [a+1]^2 E_3^{(f)} E_2 E_3 E_2 E_1^{(a+2)} \\ & = A(\mathbf{0}) + \mathbf{v}^{-a-1} \begin{bmatrix} a+1 & 0 & 1 \\ 1 & 1 & 0 \\ f & & \end{bmatrix} (\mathbf{0}) + \mathbf{v}^{-a-2} \begin{bmatrix} a+1 & 1 & 0 \\ 0 & 1 & 1 \\ f & & \end{bmatrix} (\mathbf{0}) + \mathbf{v}^{-2a-4} \begin{bmatrix} a+2 & 0 & 0 \\ 1 & 1 & 1 \\ f & & \end{bmatrix} (\mathbf{0}). \end{aligned}$$

$$(10) \quad A = \begin{bmatrix} a & 1 & 0 \\ 0 & 1 & 1 \\ f & & \end{bmatrix}, \quad E_3^{(f)} E_2 E_3 E_2 E_1 E_2 E_1^{(a)} = A(\mathbf{0}).$$

$$(11) \quad A = \begin{bmatrix} a & 1 & 1 \\ & 1 & 0 \\ & & f \end{bmatrix}, \quad \mathbf{E}_3^{(f)} \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a)} - [a] \mathbf{E}_3^{(f)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a+1)} \\ = A(\mathbf{0}) + \mathbf{v}^{-a-1} \begin{bmatrix} a & 1 & 0 \\ & 1 & 1 \\ & & f \end{bmatrix} (\mathbf{0})$$

$$(12) \quad A = \begin{bmatrix} a & 0 & 1 \\ & 1 & 1 \\ & & f \end{bmatrix}, \quad \mathbf{E}_3^{(f)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1^{(a)} + [f+2] \mathbf{E}_3^{(f+1)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a)} \\ = A(\mathbf{0}) + \mathbf{v}^{-f-1} \begin{bmatrix} a & 1 & 0 \\ & 1 & 1 \\ & & f+1 \end{bmatrix} (\mathbf{0}).$$

$$(13a) \quad A = \begin{bmatrix} a & 1 & 1 \\ & 0 & 1 \\ & & f \end{bmatrix} \text{ with } a \leq f,$$

$$\mathbf{E}_3^{(f)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a)} + [f+2] \mathbf{E}_3^{(f+1)} \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a)} \\ - [a] \mathbf{E}_3^{(f)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1^{(a+1)} - (2[a][f+2] + [f-a+1]) \mathbf{E}_3^{(f+1)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a+1)} \\ = A(\mathbf{0}) + \mathbf{v}^{-f-1} \begin{bmatrix} a & 1 & 1 \\ & 1 & 0 \\ & & f+1 \end{bmatrix} (\mathbf{0}) + \mathbf{v}^{-a-1} \begin{bmatrix} a+1 & 0 & 1 \\ & 1 & 1 \\ & & f \end{bmatrix} (\mathbf{0}) + \mathbf{v}^{-f-a-2} \begin{bmatrix} a+1 & 1 & 0 \\ & 1 & 1 \\ & & f \end{bmatrix} (\mathbf{0})$$

$$(13b) \quad A = \begin{bmatrix} a & 1 & 1 \\ & 0 & 1 \\ & & f \end{bmatrix} \text{ with } f = a-1,$$

$$\mathbf{E}_3^{(f)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a)} + [f+2] \mathbf{E}_3^{(f+1)} \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a)} \\ - [a] \mathbf{E}_3^{(f)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1^{(a+1)} - 2[a][f+2] \mathbf{E}_3^{(f+1)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a+1)} \\ = A(\mathbf{0}) + \mathbf{v}^{-f-1} \begin{bmatrix} a & 1 & 1 \\ & 1 & 0 \\ & & f+1 \end{bmatrix} (\mathbf{0}) + \mathbf{v}^{-a-1} \begin{bmatrix} a+1 & 0 & 1 \\ & 1 & 1 \\ & & f \end{bmatrix} (\mathbf{0}) + (**) \begin{bmatrix} a+1 & 1 & 0 \\ & 1 & 1 \\ & & f \end{bmatrix} (\mathbf{0})$$

$$\text{where } (**) = \mathbf{v}^{-a-1}[f+2] - \mathbf{v}^{-f-1}[a]$$

$$(13c) \quad A = \begin{bmatrix} a & 1 & 1 \\ & 0 & 1 \\ & & f \end{bmatrix} \text{ with } f \leq a-2,$$

$$\mathbf{E}_3^{(f)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a)} + [f+2] \mathbf{E}_3^{(f+1)} \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a)} \\ - [a] \mathbf{E}_3^{(f)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1^{(a+1)} - (2[a][f+2] - [a-f-1]) \mathbf{E}_3^{(f+1)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a+1)} \\ = A(\mathbf{0}) + \mathbf{v}^{-f-1} \begin{bmatrix} a & 1 & 1 \\ & 1 & 0 \\ & & f+1 \end{bmatrix} (\mathbf{0}) + \mathbf{v}^{-a-1} \begin{bmatrix} a+1 & 0 & 1 \\ & 1 & 1 \\ & & f \end{bmatrix} (\mathbf{0}) + \mathbf{v}^{-f-a-2} \begin{bmatrix} a+1 & 1 & 0 \\ & 1 & 1 \\ & & f \end{bmatrix} (\mathbf{0})$$

$$(14) \quad A = \begin{bmatrix} a & 1 & 1 \\ & 1 & 1 \\ & & f \end{bmatrix}, \quad \mathbf{E}_3^{(f)} \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_3 \mathbf{E}_2 \mathbf{E}_1 \mathbf{E}_2 \mathbf{E}_1^{(a)} = A(\mathbf{0}).$$

Proof. We just give a proof for (9). The other cases can be proved in a similar way.

For $A = \begin{bmatrix} a & 1 & 1 \\ & 0 & 1 \\ & & f \end{bmatrix} = aE_{1,2} + E_{1,3} + E_{1,4} + fE_{3,4}$, by definition,

$$\mathbf{m}_A^+ = \mathbf{E}_{3,4}^{(f)} \cdot \mathbf{E}_{1,2} \mathbf{E}_{2,3} \mathbf{E}_{3,4} \cdot \mathbf{E}_{1,2} \mathbf{E}_{2,3} \cdot \mathbf{E}_{1,2}^{(a)}.$$

Repeatedly applying the multiplication formula in Lemma 3.3 yields

$$\begin{aligned} \mathbf{m}_A^+ &= \mathbf{E}_{3,4}^{(f)} ((aE_{1,2} + E_{1,3} + E_{1,4})(\mathbf{0}) \\ &\quad + \mathbf{v}^{a-1} \overline{[a+1]} ((a+1)E_{1,2} + E_{2,3} + E_{1,4})(\mathbf{0}) \\ &\quad + (\mathbf{v}^{a-2} \overline{[a+1]} + \mathbf{v}^a \overline{[a+1]}) ((a+1)E_{1,2} + E_{1,3} + E_{2,4})(\mathbf{0}) \\ &\quad + \mathbf{v}^{a-1} \overline{[a+1]} \mathbf{v}^{a-1} \overline{[a+2]} ((a+2)E_{1,2} + E_{2,3} + E_{2,4})(\mathbf{0}) \\ &\quad + \mathbf{v}^{a+1} \overline{[a+1]} [f+1] ((a+1)E_{1,2} + E_{2,3} + E_{1,3} + E_{3,4})(\mathbf{0})) \end{aligned} \quad (6.3.1)$$

For $C = (c_{i,j}) \in M(2|2)^+$, by the multiplication formula in Lemma 3.4,

$$E_{3,4}^{(f)}C(\mathbf{0}) = \mathbf{v}_3^{fc_{3,4}} \overline{\begin{bmatrix} f + c_{3,4} \\ f \end{bmatrix}}_{\mathbf{v}_3} (C + fE_{3,4})(\mathbf{0}).$$

Then, noting $\mathbf{v}_3^f \overline{\begin{bmatrix} f+1 \\ f \end{bmatrix}}_{\mathbf{v}_3} = [f+1]_{\mathbf{v}_3} = [f+1]$,

$$E_{3,4}^{(f)}C(\mathbf{0}) = \begin{cases} (C + fE_{3,4})(\mathbf{0}), & \text{if } c_{3,4} = 0; \\ [f+1](C + fE_{3,4})(\mathbf{0}), & \text{if } c_{3,4} = 1. \end{cases}$$

Thus,

$$\begin{aligned} \mathbf{m}_A^+ &= A(\mathbf{0}) + \mathbf{v}^{a-1} \overline{[a+1]}((a+1)E_{1,2} + E_{2,3} + E_{1,4} + fE_{3,4})(\mathbf{0}) \\ &\quad + (\mathbf{v}^{a-2} \overline{[a+1]} + \mathbf{v}^a \overline{[a+1]})((a+1)E_{1,2} + E_{1,3} + E_{2,4} + fE_{3,4})(\mathbf{0}) \\ &\quad + \mathbf{v}^{a-1} \overline{[a+1]} \mathbf{v}^{a-1} \overline{[a+2]}((a+2)E_{1,2} + E_{2,3} + E_{2,4} + fE_{3,4})(\mathbf{0}) \\ &\quad + \mathbf{v}^{a+1} \overline{[a+1]}[f+1]((a+1)E_{1,2} + E_{2,3} + E_{1,3} + (f+1)E_{3,4})(\mathbf{0}) \end{aligned} \quad (6.3.2)$$

Observing the summands above, the maximal matrix B such that $B \prec A$ is $B_1 = (a+1)E_{1,2} + E_{2,3} + E_{1,4} + fE_{3,4}$, and the coefficient of $B_1(\mathbf{0})$ is

$$\mathbf{v}^{a-1} \overline{[a+1]} = [a] + \mathbf{v}^{-a-1}.$$

So we compute $\mathbf{m}_A^+ - [a]\mathbf{m}_{B_1}^+$. By the multiplication formulas, we have

$$\begin{aligned} \mathbf{m}_{B_1}^+ &= E_{3,4}^{(f)}E_{1,2}E_{2,3}E_{3,4}E_{2,3}E_{1,2}^{(a+1)} \\ &= B_1(\mathbf{0}) + \mathbf{v}^{-1}((a+1)E_{1,2} + E_{1,3} + E_{2,4} + fE_{3,4})(\mathbf{0}) \\ &\quad + \mathbf{v}^{a-1} \overline{[a+2]}((a+2)E_{1,2} + E_{2,3} + E_{2,4} + fE_{3,4})(\mathbf{0}). \end{aligned}$$

Then

$$\begin{aligned} \mathbf{m}_A^+ - [a]\mathbf{m}_{B_1}^+ &= A(\mathbf{0}) + \mathbf{v}^{-a-1}B_1(\mathbf{0}) \\ &\quad + (\mathbf{v}^{a-2} \overline{[a+1]} + \mathbf{v}^a \overline{[a+1]} - \mathbf{v}^{-1}[a])((a+1)E_{1,2} + E_{1,3} + E_{2,4} + fE_{3,4})(\mathbf{0}) \\ &\quad + (\mathbf{v}^{a-1} \overline{[a+1]} \mathbf{v}^{a-1} \overline{[a+2]} - \mathbf{v}^{a-1} \overline{[a+2]}[a])((a+2)E_{1,2} + E_{2,3} + E_{2,4} + fE_{3,4})(\mathbf{0}) \\ &\quad + \mathbf{v}^{a+1} \overline{[a+1]}[f+1]((a+1)E_{1,2} + E_{2,3} + E_{1,3} + (f+1)E_{3,4})(\mathbf{0}) \end{aligned} \quad (6.3.3)$$

Now the maximal matrices B in (6.3.3) such that $B \prec B_1$ is

$$B_2 = (a+1)E_{1,2} + E_{1,3} + E_{2,4} + fE_{3,4}.$$

Since the coefficient of $B_2(\mathbf{0})$ in (6.3.3) is

$$\begin{aligned} &\mathbf{v}^{a-2} \overline{[a+1]} + \mathbf{v}^a \overline{[a+1]} - \mathbf{v}^{-1}[a] \\ &= [a-1] + \mathbf{v}^{-a} + \mathbf{v}^{-a-2} + [a+1] - [a-1] - \mathbf{v}^{-a} = \mathbf{v}^{-a-2} + [a+1], \end{aligned}$$

We now compute $\mathbf{m}_A^+ - [a]\mathbf{m}_{B_1}^+ - [a+1]\mathbf{m}_{B_2}^+$. Since

$$\begin{aligned}\mathbf{m}_{B_2}^+ &= \mathbf{E}_{3,4}^{(f)} \mathbf{E}_{2,3} \mathbf{E}_{3,4} \mathbf{E}_{1,2} \mathbf{E}_{2,3} \mathbf{E}_{1,2}^{(a+1)} \\ &= B_2(\mathbf{0}) + \mathbf{v}^a \overline{[a+2]}((a+2)E_{1,2} + E_{2,3} + E_{2,4} + fE_{3,4})(\mathbf{0}) \\ &\quad + \mathbf{v}[f+1]((a+1)E_{1,2} + E_{2,3} + E_{1,3} + (f+1)E_{3,4})(\mathbf{0}),\end{aligned}$$

it yields

$$\begin{aligned}\mathbf{m}_A^+ - [a]\mathbf{m}_{B_1}^+ - [a+1]\mathbf{m}_{B_2}^+ &= A(\mathbf{0}) + \mathbf{v}^{-a-1}B_1(\mathbf{0}) + \mathbf{v}^{-a-2}B_2(\mathbf{0}) \\ &\quad + (\mathbf{v}^{a-1}\overline{[a+1]}\mathbf{v}^{a-1}\overline{[a+2]} - \mathbf{v}^{a-1}\overline{[a+2]}[a] - \mathbf{v}^a[a+1]\overline{[a+2]}) \\ &\quad ((a+2)E_{1,2} + E_{2,3} + E_{2,4} + fE_{3,4})(\mathbf{0}) \\ &\quad + (\mathbf{v}^{a+1}\overline{[a+1]}[f+1] - \mathbf{v}[a+1][f+1])((a+1)E_{1,2} + E_{2,3} + E_{1,3} + (f+1)E_{3,4})(\mathbf{0})\end{aligned}$$

But the coefficient of $((a+1)E_{1,2} + E_{2,3} + E_{1,3} + (f+1)E_{3,4})(\mathbf{0})$ is

$$\mathbf{v}^{a+1}\overline{[a+1]}[f+1] - [a+1]\mathbf{v}[f+1] = \mathbf{v}[a+1][f+1] - \mathbf{v}[a+1][f+1] = 0,$$

so

$$\begin{aligned}\mathbf{m}_A^+ - [a]\mathbf{m}_{B_1}^+ - [a+1]\mathbf{m}_{B_2}^+ &= A(\mathbf{0}) + \mathbf{v}^{-a-1}B_1(\mathbf{0}) + \mathbf{v}^{-a-2}B_2(\mathbf{0}) \\ &\quad + (\mathbf{v}^{a-1}\overline{[a+1]}\mathbf{v}^{a-1}\overline{[a+2]} - \mathbf{v}^{a-1}\overline{[a+2]}[a] - \mathbf{v}^a[a+1]\overline{[a+2]}) \\ &\quad ((a+2)E_{1,2} + E_{2,3} + E_{2,4} + fE_{3,4})(\mathbf{0})\end{aligned}\tag{6.3.4}$$

Let $B_3 = (a+2)E_{1,2} + E_{2,3} + E_{2,4} + fE_{3,4}$ and rewrite coefficient of $B_3(\mathbf{0})$ as

$$\begin{aligned}\mathbf{v}^{a-1}\overline{[a+1]}\mathbf{v}^{a-1}\overline{[a+2]} - \mathbf{v}^{a-1}\overline{[a+2]}[a] - \mathbf{v}^a[a+1]\overline{[a+2]} \\ = \mathbf{v}^{-a-1}[a] - \mathbf{v}^{-a-2}[a+1] + \mathbf{v}^{-2a-2} + \mathbf{v}^{-2a-4} - [a+1]^2 \\ = \mathbf{v}^{-2a-4} - [a+1]^2.\end{aligned}$$

Finally, we compute $\mathbf{m}_A^+ - [a]\mathbf{m}_{B_1}^+ - [a+1]\mathbf{m}_{B_2}^+ + [a+1]^2\mathbf{m}_{B_3}^+$. Since

$$\mathbf{m}_{B_3}^+ = \mathbf{E}_{3,4}^{(f)} \mathbf{E}_{2,3} \mathbf{E}_{3,4} \mathbf{E}_{2,3} \mathbf{E}_{1,2}^{(a+2)} = B_3(\mathbf{0}),$$

it follows that

$$\begin{aligned}\mathbf{m}_A^+ - [a]\mathbf{m}_{B_1}^+ - [a+1]\mathbf{m}_{B_2}^+ + [a+1]^2\mathbf{m}_{B_3}^+ \\ = A(\mathbf{0}) + \mathbf{v}^{-a-1}B_1(\mathbf{0}) + \mathbf{v}^{-a-2}B_2(\mathbf{0}) + \mathbf{v}^{-2a-4}B_3(\mathbf{0})\end{aligned}$$

is the required canonical basis element \mathbf{C}_A . □

7. SIMPLE POLYNOMIAL REPRESENTATIONS OF $\mathbf{U}(\mathfrak{gl}_{m|n})$

For a finite dimensional $\mathbf{U}(\mathfrak{gl}_{m|n})$ -module M and $\lambda \in \mathbb{Z}^{m+n}$, let

$$M_\lambda = \left\{ x \in M \mid K_i x = \mathbf{v}_i^{\lambda_i} x, 1 \leq i \leq m+n \right\}.$$

If $M_\lambda \neq 0$, then M_λ is called the weight space of M of weight λ . Call M an *integral weight module* (of type **1**) if $M = \bigoplus_\lambda M_\lambda$ and denote by $\text{wt}(M)$ the set of all weights of M . A weight module M is called a *polynomial representation* of $\mathbf{U}(\mathfrak{gl}_{m|n})$ if $\text{wt}(M) \subset \mathbb{N}^{m+n}$. Clearly, a tensor power of a polynomial representation is polynomial. In

particular, the tensor power $V^{\otimes r}$ of the natural representation V of $U(\mathfrak{gl}_{m|n})$ is a polynomial representation.

Let $\mathbf{U} = \mathbf{U}(\mathfrak{gl}_{m|n})$ and $\mathbf{U}_{\bar{0}} = \mathbf{U}(\mathfrak{gl}_m \oplus \mathfrak{gl}_n)$ and $\mathbf{U}_{\bar{1}}^{\pm} = \mathbf{U}(\mathfrak{gl}_{m|n, \bar{1}}^{\pm})$. For $\lambda \in \Lambda^+(m|n)$, let $L^{\bar{0}}(\lambda)$ be the (finite dimensional) irreducible module of $\mathbf{U}_{\bar{0}}$ with the highest weight λ . Then $L^{\bar{0}}(\lambda)$ becomes a module of the parabolic superalgebra $\mathbf{U}_{\bar{0}}\mathbf{U}_{\bar{1}}^+$ via the trivial action of $E_{a,b}$ on $L^{\bar{0}}(\lambda)$ for all $1 \leq a \leq m < b \leq m+n$. Define the *Kac-module* (see [18])

$$K(\lambda) = \text{Ind}_{\mathbf{U}_{\bar{0}}\mathbf{U}_{\bar{1}}^+}^{\mathbf{U}} L^{\bar{0}}(\lambda) = \mathbf{U} \otimes_{\mathbf{U}_{\bar{0}}\mathbf{U}_{\bar{1}}^+} L^{\bar{0}}(\lambda).$$

Since \mathbf{U} is a free $\mathbf{U}_{\bar{0}}\mathbf{U}_{\bar{1}}^+$ module, as vector spaces, we have

$$K(\lambda) \cong \mathbf{U}_{\bar{1}}^- \otimes L^{\bar{0}}(\lambda).$$

Note that, for all $\mu \in \text{wt}(K(\lambda))$, $|\lambda| = |\mu|$ and $\mu \leq \lambda$ (meaning $\lambda - \mu$ is a sum of positive roots).⁴ Thus, we say that $K(\lambda)$ is a representation of \mathbf{U} at *level* $|\lambda|$. Moreover, every $K(\lambda)$ has a unique maximal submodule and hence defines a simple module $L(\lambda)$. In fact, the set $\{L(\lambda) \mid \lambda \in \Lambda^+(m|n)\}$ forms a complete set of finite dimensional simple \mathbf{U} -modules.

Since every irreducible finite dimensional module $L(\lambda)$ of \mathbf{U} is a quotient module of a Kac module $K(\lambda)$, $L(\lambda)$ is a representation at the same level as $K(\lambda)$.

Lemma 7.1. *The irreducible polynomial representations of $\mathbf{U}(\mathfrak{gl}_{m|n})$ at level $r \geq 0$ are all inflated via η_r from the irreducible representations of $\mathcal{S}(m|n, r)$.*

Proof. Clearly, if M is an $\mathcal{S}(m|n, r)$ -module, then $M = \bigoplus_{\lambda \in \Lambda(m|n, r)} M_{\lambda}$ as a $\mathbf{U}(\mathfrak{gl}_{m|n})$ -module, where $M_{\lambda} = \xi_{\lambda} M$ with $\xi_{\lambda} = [\text{diag}(\lambda)]$. This is seen easily since $\eta_r(K_i) = \sum_{\lambda} \mathbf{v}_i^{\lambda_i} \xi_{\lambda}$. Hence, every inflated module is a module at level r .

Assume now M is an irreducible polynomial representation of $\mathbf{U}(\mathfrak{gl}_{m|n})$ at level r . For any $0 \neq x \in M_{\mu}$,

$$\begin{aligned} K_1 K_2 \cdots K_m K_{m+1}^{-1} \cdots K_{m+n}^{-1} \cdot x &= \mathbf{v}^{\sum_{i=1}^{m+n} \mu_i} x = \mathbf{v}^r x \\ (K_i - 1)(K_i - \mathbf{v}_i) \cdots (K_i - \mathbf{v}_i^r) \cdot x &= \prod_{j=0}^r (\mathbf{v}_i^{\mu_i^j} - \mathbf{v}_i^j) x = 0x = 0. \end{aligned}$$

By the presentation for $\mathcal{S}(m|n, r)$ given in [10], we see that M is in fact an inflation of a simple $\mathcal{S}(m|n, r)$ -module. \square

By this lemma, the study of simple polynomial representations of $\mathbf{U}(\mathfrak{gl}_{m|n})$ is reduced to that of simple $\mathcal{S}(m|n, r)$ -modules for all $r \geq 0$. Simple $\mathcal{S}(m|n, r)$ -modules have been classified and constructed in [8] via a certain cellular basis. We now use the cellular bases adjusted with a sign as in defining the canonical basis $\{\Xi_A\}_A$ to see how the canonical bases for $U_{\mathcal{Z}}^-$ and $\mathcal{S}(m|n, r)$ induce related bases for these modules.

⁴Since $\lambda - \mu = (\lambda_1 - \mu_1)\mathbf{e}_1 + \cdots + (\lambda_n - \mu_n)\mathbf{e}_n = (\tilde{\lambda}_1 - \tilde{\mu}_1)(\mathbf{e}_1 - \mathbf{e}_2) + \cdots + (\tilde{\lambda}_{n-1} - \tilde{\mu}_{n-1})(\mathbf{e}_{n-1} - \mathbf{e}_n)$ ($\tilde{a}_j = \sum_{i=1}^j a_i$), this order is the usual dominance order \supseteq if λ, μ are regarded as compositions.

For $A = j(\lambda, d, \mu)$ as in (2.0.3), define compositions α, β by (2.0.5),

$$\mathfrak{S}_{\alpha|\beta} := \mathfrak{S}_{\lambda d \cap \mu} \cong (\mathfrak{S}_{\lambda^{(0)}}^d \cap \mathfrak{S}_{\mu^{(0)}}) \times (\mathfrak{S}_{\lambda^{(1)}}^d \cap \mathfrak{S}_{\mu^{(1)}}). \quad (7.1.1)$$

Let $\xi'_A = \mathbf{v}^{l(w_0, \beta)} P_{\mathfrak{S}_\beta}(\mathbf{v}^{-2}) \xi_A$ (cf. [8, (6.3.1)]). Using this basis and the bar involution defined in (2.2.1) (cf. [8, Th. 6.3]), one defines another canonical basis $\{\Xi'_A \mid A \in M(m|n, r)\}$ for $\mathcal{S}(m|n, r)$. Note that this basis is not integral basis over \mathbb{Z} but a cellular basis over $\mathbb{Q}(\mathbf{v})$. We now describe its cellularity.

For $\lambda \in \Lambda(m|n, r), \mu \in \Lambda(m'|n', r)$, let

$$\mathcal{D}_{\lambda, \mu}^{+, -} = \mathcal{D}_{(\lambda^{(0)}|1^{a_1}), (\mu^{(0)}|1^{b_1})}^+ \cap \mathcal{D}_{(1^{a_0}|\lambda^{(1)}), (1^{b_0}|\mu^{(1)})}, \quad (\text{see [8, (3.0.4)]})$$

where $a_i = |\lambda^{(i)}|$ and $b_i = |\mu^{(i)}|$. Then the map (2.0.3) induces a bijection

$$j^{+, -} : \mathcal{D}(m|n, r)^{+, -} \longrightarrow M(m|n, r),$$

where

$$\mathcal{D}(m|n, r)^{+, -} = \{(\lambda, w, \mu) \mid \lambda, \mu \in \Lambda(m|n, r), w \in \mathcal{D}_{\lambda, \mu}^{+, -}\}.$$

Definition 7.2. Let $A = j^{+, -}(\alpha, y, \beta), B = j^{+, -}(\lambda, w, \mu) \in M(m|n, r)$. Define

$$A \leq_L B \iff y \leq_L w \text{ and } \mu = \beta,$$

where $y \leq_L w$ is the order relation \leq_L on \mathfrak{S}_r defined in [12].

With this order relation, the structure constants for the basis $\{\Xi'_A\}_{A \in M(m|n, r)}$ satisfy the following order relation.

Lemma 7.3 ([8, 7.2]). *For $A, B \in M(m|n, r)$, if $\Xi'_A \Xi'_B = \sum_{C \in M(m|n, r)} f_{A, B, C} \Xi'_C$, then $f_{A, B, C} \neq 0$ implies $C \leq_L B$ and $C \leq_R A$.*

Define $A \leq_R B$ if $A^t \leq_L B^t$. Let \leq_{LR} be the preorder generated by \leq_L and \leq_R . The relations give rise to three equivalence relations \sim_L, \sim_R and \sim_{LR} on $M(m|n, r)$. Thus, $A \sim_X B$ if and only if $A \leq_X B \leq_X A$ for all $X \in \{L, R, LR\}$. The corresponding equivalence classes in $M(m|n, r)$ with respect to \sim_L, \sim_R and \sim_{LR} are called *left cells*, *right cells* and *two-sided cells* respectively.

Like the symmetric group case, the cells defined here can also be described in terms of a super version of the Robinson–Schensted correspondence.

Let $\Pi(r)$ be the set of all partitions of r and let

$$\Pi(r)_{m|n} = \{\pi \in \Pi(r) \mid \pi_{m+1} \leq n\}.$$

For $\pi \in \Pi(r)_{m|n}$ and $\mu \in \Lambda(m|n, r)$. A π -tableau \mathbb{T} of content μ is called a *semi-standard π -supertableau* of type μ if, in addition,

- a) the entries are weakly increasing in each row and each column of \mathbb{T} ;
- b) the numbers in $\{1, 2, \dots, m\}$ are strictly increasing in the columns and the numbers in $\{m+1, m+2, \dots, m+n\}$ are strictly increasing in the rows.

Let $\mathbf{T}^{su}(\pi, \mu)$ be the set of all semi-standard π -supertableaux of content μ . In particular, for the given partition π , if we set

$$\tilde{\pi}^{(0)} = (\pi_1, \pi_2, \dots, \pi_m), \quad \tilde{\pi}^{(1)} = (\pi_{m+1}, \pi_{m+2}, \dots, \pi_{m+n})^t, \quad (7.3.1)$$

then $\tilde{\pi} = (\tilde{\pi}^{(0)} | \tilde{\pi}^{(1)}) \in \Lambda(m|n, r)$ and $\mathbf{T}^{su}(\pi, \tilde{\pi})$ contains a unique element, denoted by \mathbf{T}_π . We also write $\text{sh}(\mathbf{T}) = \pi$ if $\mathbf{T} \in \mathbf{T}^{su}(\pi, \mu)$, called the *shape* of \mathbf{T} .

Lemma 7.4 ([8, 4.7, 7.3]). *There is a bijective map*

$$\text{RSKs} : M(m|n, r) \longrightarrow \bigcup_{\substack{\lambda, \mu \in \Lambda(m|n, r) \\ \pi \in \Pi(r)_{m|n}}} \mathbf{T}^{su}(\pi, \lambda) \times \mathbf{T}^{su}(\pi, \mu), \quad A \longrightarrow (\mathbf{p}(A), \mathbf{q}(A))$$

such that $\lambda = \text{ro}(A)$, $\mu = \text{co}(A)$ and, for $A, B \in M(m|n, r)$ with $\pi^t = \text{sh}(\mathbf{p}(A))$, $\nu^t = \text{sh}(\mathbf{p}(B))$,

- (1) $A \sim_L B$ if and only if $\mathbf{q}(A) = \mathbf{q}(B)$.
- (2) $A \sim_R B$ if and only if $\mathbf{p}(A) = \mathbf{p}(B)$.
- (3) $A \leq_{LR} B$ implies $\pi \supseteq \nu$. Hence, $A \sim_{LR} B$ if and only if $\mathbf{p}(A), \mathbf{p}(B)$ have the same shape.

For $\pi \in \Pi(r)_{m|n}$, let

$$I(\pi) = \bigcup_{\lambda \in \Lambda(m|n, r)} \mathbf{T}^{su}(\pi, \lambda).$$

By the super RSK correspondence, if $A \xrightarrow{\text{RSKs}} (\mathbf{S}, \mathbf{T}) \in I(\pi)$, we relabel the basis element Ξ'_A as $\Xi'_{\mathbf{S}, \mathbf{T}} := \Xi'_A$.

Lemma 7.5 ([8, 7.4]). *The $\mathbb{Q}(\mathbf{v})$ -basis for $\mathcal{S}(m|n, r)$*

$$\{\Xi'_{\mathbf{S}, \mathbf{T}} \mid \pi \in \Lambda^+(r)_{m|n}, \mathbf{S}, \mathbf{T} \in I(\pi)\} = \{\Xi'_A \mid A \in M(m|n, r)\}.$$

is a cellular basis in the sense of [11].

The cellular basis defines cell modules $C(\pi)$, $\pi \in \Pi(r)_{m|n}$ (see [11] or [6, (C.6.3)]). Since $\mathcal{S}(m|n, r)$ is semisimple, all $C(\pi)$ are irreducible.

Theorem 7.6. *As a $U(\mathfrak{gl}_{m|n})$ -module via $\eta_r : U(\mathfrak{gl}_{m|n}) \rightarrow \mathcal{S}(m|n, r)$, $C(\pi) \cong L(\tilde{\pi})$, where $\tilde{\pi}$ is defined in (7.3.1).*

Proof. By the construction, for any fixed $\mathbf{Q} \in I(\pi)$, $C(\pi)$ is spanned by $v_{\mathbf{S}} := \Xi'_{\mathbf{S}, \mathbf{Q}} + \mathcal{S}^{\triangleright \pi}$, $\mathbf{S} \in I(\pi)$, where $\mathcal{S}^{\triangleright \pi}$ is spanned by all Ξ'_A with $\text{sh}(\mathbf{p}(A)) \triangleright \pi$. Let $v_{\tilde{\pi}} = v_{\mathbf{T}_\pi}$. Then the weight of $v_{\tilde{\pi}}$ is $\tilde{\pi}$. We now prove that $\tilde{\pi}$ is the highest weight. It suffices to prove that if $\mathbf{T}^{su}(\pi, \mu) \neq \emptyset$ then $\tilde{\pi} \supseteq \mu$.

Let $\mathbf{T} \in \mathbf{T}^{su}(\pi, \mu)$ and, for $s \in [1, m+n]$, let $\mathbf{T}_{\leq s}$ be the subtableau obtained by removing the entries $> s$ and their associated boxes from \mathbf{T} . If $s \leq m$, then it is known that $\tilde{\pi}_1 + \dots + \tilde{\pi}_s \geq \mu_1 + \dots + \mu_s$ (see, e.g., [6, Lem. 8.42]). Assume now $s > m$. Let $\mathbf{T}'_{\leq s}$ be the subtableau consists of top m row of $\mathbf{T}_{\leq s}$ and $\mathbf{T}''_{\leq s}$ the subtableau obtained by removing $\mathbf{T}'_{\leq s}$ from $\mathbf{T}_{\leq s}$. We also break \mathbf{T}_π into two parts $\mathbf{T}'_{\pi, \leq s}$ and $\mathbf{T}''_{\pi, \leq s}$. Then,

by definition, the shape of $\mathbf{T}'_{\leq s}$ must be contained in $\mathbf{T}'_{\pi, \leq s}$, while the shape of $\mathbf{T}''_{\leq s}$ must be contained in $\mathbf{T}''_{\pi, \leq s}$. Hence, $\tilde{\pi}_1 + \cdots + \tilde{\pi}_s \geq \mu_1 + \cdots + \mu_s$. This proves the inequality for all $s \geq 0$. Hence, $\tilde{\pi} \supseteq \mu$. \square

Remark 7.7. Unlike the nonsuper case, the cellular basis $\{\Xi'_A \mid A \in M(m|n, r)\}$ does not canonically induce a basis for $C(\pi)$. In other words, the set $\{\Xi'_A \cdot v_{\tilde{\pi}} \mid A \in M(m|n, r)\} \setminus \{0\}$ does not form a basis for the cell module $C(\pi)$. This can be seen as follows. Suppose $\Xi'_A = \Xi''_{S, \mathbf{T}}$. Then

$$0 \neq \Xi'_A \cdot v_{\tilde{\pi}} = \Xi''_{S, \mathbf{T}} \Xi'_{\mathbf{T}, \mathbf{Q}} \pi + \mathcal{S}^{\triangleright \pi} = \sum_C f_C(A, \mathbf{T}_\pi) \Xi'_C \pi + \mathcal{S}^{\triangleright \pi}$$

implies $\text{co}(A) = \tilde{\pi}$, $\tilde{\nu} \supseteq \tilde{\pi}$ by the proof above, and $C \leq_R A$ by Lemma 7.3. Hence, $\pi \supseteq \nu$ by Lemma 7.4(3). Thus, we must have $\tilde{\pi}^{(0)} = \tilde{\nu}^{(0)}$ and $\tilde{\pi}^{(1)} \leq \tilde{\nu}^{(1)}$ (equivalently, $\pi^{(1)} \supseteq \nu^{(1)}$). Cosequently, we do not have $\nu = \pi$ in general unless $n = 1$ and so the cardinality of the set could be larger than $\dim C(\pi)$.

Corollary 7.8. *We have $\mathcal{S}(m|n, r)^+ v_{\tilde{\pi}} = 0$. In other words, by regarding $C(\pi)$ as a $U(\mathfrak{gl}_{m|n})$ -module, $v_{\tilde{\pi}}$ is a primitive vector.*

Proof. We first observe that, if $\text{co}(E_{h, h+1} + \text{diag}(\lambda)) = \tilde{\pi}$, i.e., $\lambda + \mathbf{e}_{h+1} = \tilde{\pi}$, then $\lambda = \tilde{\pi} - \mathbf{e}_{h+1}$. Since $E_{h, h+1}(\mathbf{0}, r) \cdot v_{\tilde{\pi}} = [E_{h, h+1} + \text{diag}(\lambda)]v_{\tilde{\pi}}$ has weight $\tilde{\pi} + \mathbf{e}_h - \mathbf{e}_{h+1}$ and $\tilde{\pi} + \mathbf{e}_h - \mathbf{e}_{h+1} \triangleright \tilde{\pi}$, we must have $E_{h, h+1}(\mathbf{0}, r) \cdot v_{\tilde{\pi}} = 0$ by the theorem above. \square

Corollary 7.9. *Let $M(m|n, r)^{\leq 0} = \{(a_{i,j}) \in M(m|n, r) \mid a_{i,j} = 0 \forall i < j\}$. Then the set $\{\Xi_A \cdot v_{\tilde{\pi}} \mid A \in M(m|n, r)^{\leq 0}\}$ spans the cell module $C(\pi)$.*

It would be interesting to extract a basis for $C(\pi)$ from this spanning set in some “canonical” way. This is because such a basis can also be induced from the canonical basis \mathcal{C}^- of $U_{\mathcal{Z}}^-(\mathfrak{gl}_{m|n})$ as shown in the following result.

For $\pi \in \Pi(r)_{m|n}$, define a subset $M(m|n, \pi)$ of $M(m|n, r)^{\leq 0}$ by the condition that

$$\{\Xi_A \cdot v_{\tilde{\pi}} \mid A \in M(m|n, \pi)\} \text{ forms a basis for } C(\pi).$$

Note that $\text{co}(A) = \tilde{\pi}$ for $A \in M(m|n, \pi)$. Recall the notation A_λ defined in (5.3.2).

Theorem 7.10. *Let \mathcal{C}^- be the canonical basis for $U_{\mathcal{Z}}^-(\mathfrak{gl}_{m|n})$ and let $L(\mu)$ be a simple polynomial representations of $U(\mathfrak{gl}_{m|n})$ at level $r \geq 0$. Then there exists a partition $\pi \in \Pi(r)_{m|n}$ such that $\mu = \tilde{\pi}$ and*

$$\{\mathbf{C}_A \cdot v_{\tilde{\pi}} \mid A \in M(m|n)^-, A_{\tilde{\pi}} \in M(m|n, \pi)\}$$

forms a basis for $L(\mu)$.

Proof. The first assertion follows from Lemma 7.1 and Theorem 7.6. Thus, $L(\mu) \cong C(\pi)$. By Theorem 5.4,

$$\mathbf{C}_A \cdot v_{\tilde{\pi}} = \mathbf{c}_A \cdot v_{\tilde{\pi}} = \sum_{\lambda: \lambda \geq \mathfrak{h}(A)} (-1)^{\bar{A}_\lambda} \Xi_{A_\lambda} \cdot v_{\tilde{\pi}} = (-1)^{\bar{A}_{\tilde{\pi}}} \Xi_{A_{\tilde{\pi}}} \cdot v_{\tilde{\pi}}.$$

The last assertion follows from the definition of $M(m|n, \pi)$. \square

We end the paper with a canonical description of $M(m|n, \pi)$ for $n = 1$. This case has already been considered in [5] is a natural application of the observation in Remark 7.7.

Theorem 7.11. *For $\pi \in \Pi(r)_{m|1}$, the set $\{\Xi_A \cdot v_\pi \mid A \in M(m|1, r)\} \setminus \{0\}$ forms a basis for the cell module $C(\pi)$.*

Proof. We first claim that the set $\{\Xi'_A \cdot v_\pi \mid A \in M(m|1, r)\} \setminus \{0\}$ forms a basis for $C(\pi)$. Indeed, assume $\Xi'_A = \Xi''_{S, T}$. By Remark 7.7, $\Xi'_A \cdot v_{\tilde{\pi}} \neq 0$ implies $\text{co}(A) = \tilde{\pi}$, $\pi \succeq \nu$ and $\tilde{\pi}^{(0)} = \tilde{\nu}^{(0)}$. Since $|\pi| = |\nu|$ and $n = 1$, we must have $\pi_{m+1} = \nu_{m+1}$, forcing $\pi = \nu$ and $T = T_\pi$. Thus, a dimensional comparison proves the claim.

Further, for $\pi \in \Pi(r)_{m|1}$, $\pi_{m+1} \leq 1$. This forces the subgroup \mathfrak{S}_β defined in (7.1.1) is trivial. Hence, $\xi'_A = \xi_A$. By the argument given around [8, Rem. 6.5], we also have $\Xi'_A = \Xi_A$ whenever $\text{co}(A) = \tilde{\pi}$. Now the result follows from the claim above. \square

With this theorem, the index set $M(m|1, \pi)$ can have the following canonical description:

$$\begin{aligned} M(m|1, \pi) &= \{A \in M(m|1, r) \mid \Xi'_A \cdot v_\pi \neq 0\} \\ &= \{A \in M(m|1, r) \mid \mathfrak{p}(A) \in I(\pi), \mathfrak{q}(A) = T_\pi\}. \end{aligned}$$

Theorems 7.10 and 7.11 gives immediately the following.

Corollary 7.12. *Let $\mathcal{C}^- = \{C_A \mid A \in M(m|1)^-\}$ be the canonical basis for $U_{\mathbb{Z}}^-(\mathfrak{gl}_{m|1})$ as given in (5.2.1) and let $L(\mu)$ be a simple polynomial representations of $U(\mathfrak{gl}_{m|1})$ with highest weight vector v_μ . Then*

$$\{C_A \cdot v_\mu \mid A \in M(m|1)^-\} \setminus \{0\}$$

forms a basis for $L(\mu)$.

We have proved the conjecture [5, Conj. 8.9] for polynomial representations.

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